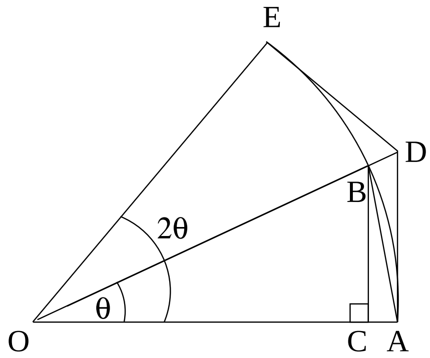
Consider a circular sector with center O and radius 1. Let A, B , E be points on the circular sector as shown.




Then:

$$\text{length}(BC) = \sin(\theta), \qquad \text{length}(\text{Arc(AB)}) = \theta, \qquad \text{and} \qquad \text{length}(AD) = \tan(\theta)\ .$$

### 14.1 1st inequalties and the limit $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

For $0 < \theta < \frac{\pi}{4}$ , so $2\theta$ is at most $\frac{\pi}{2}$ (90 degrees), we have:

$$\text{length}(BC) < \text{length}(AB) < \text{length}(\text{Arc(AB)}) \ , \ \text{so} \ \sin(\theta) < \text{length}(AB) < \theta$$

and

$$\text{length}(\text{Arc(AB)}) < \text{length}(AD) \ , \ \text{so} \ \theta < \tan(\theta).$$

In summary, we have:

$$\sin(\theta) < \theta < \tan(\theta)$$

We manipulate to:

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1 \ , \ \text{valid for} \ \ 0 < \theta < \frac{\pi}{4}.$$

The three functions $\cos(\theta)$, $\frac{\sin(\theta)}{\theta}$, and 1 are all even, so the inequality is also true for $-\frac{\pi}{4} < \theta < 0$. In a picture:


The rule $\dfrac{\sin(\theta)}{\theta}$ does NOT allow input of $\theta = 0$. Zero is not in the domain. But, the function $\dfrac{\sin(\theta)}{\theta}$ is caught (for $\theta \neq 0$) between the two functions $\cos(\theta)$ and 1. We can apply the squeeze theorem to get

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \quad = \; 1 \, .$$

### 14.2 2nd limit involving the function $\frac{1-\cos(\theta)}{\theta}$

From the 1st picture we have:

$$\text{length}(AC)^2 \; + \; \text{length}(BC)^2 \; = \; \text{length}(AB)^2 \; \leq \; \text{length}(\text{Arc(AB)})^2 \; , \quad \text{so}$$

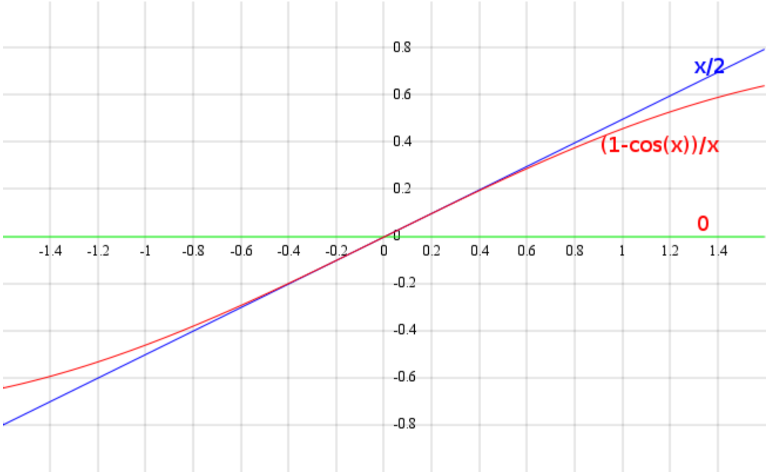
$$(1 - \cos(\theta))^2 \; + \; (\sin(\theta))^2 \; \leq \; \theta^2 \; , \qquad \text{then expand to get}$$

$$2 \, ( \, 1 \; - \; \cos(\theta) \, ) \; \leq \; \theta^2 \; , \qquad \text{and deduce}$$

$$0 \; \leq \; \frac{( \, 1 \; - \; \cos(\theta) \, )}{\theta} \; \leq \; \frac{\theta}{2} \, , \qquad \text{for } 0 \; < \theta \; < \frac{\pi}{4}$$

The (odd symmetry) rule $\dfrac{( \, 1 \; - \; \cos(\theta) \, )}{\theta}$ does not allow input $\theta = 0$.

A picture of this rule (graph in red), with values of $\frac{(1 - \cos(\theta))}{\theta}$ caught between 0 (graph in yellow) and $\frac{\theta}{2}$ (graph in blue), is:




As before, we can deduce

$$\lim_{\theta \to 0} \frac{(1 - \cos(\theta))}{\theta} = 0 .$$

These two limits $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$, and $\lim_{\theta \to 0} \frac{(1 - \cos(\theta))}{\theta} = 0$ are very important. They are used later to compute the slope of tangent lines to the functions sin and cos.

## 15 Tangent slope of the functions sine and cosine

### 15.1 Tangent slope to the graph of sine at the point $(b, \sin(b))$.

The tangent slope at the graph point $(b, \sin(b))$ is the limit of the difference quotient:

$$\frac{\sin(b + h) - \sin(b)}{h} .$$

We use the formula for the sine of a sum to get:

$$\begin{aligned} \sin(b + h) - \sin(b) &= \sin(b) \cos(h) + \cos(b) \sin(h) - \sin(b) \\ &= \sin(b) (\cos(h) - 1) + \cos(b) \sin(h) \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0} \frac{\sin(b + h) - \sin(b)}{h} &= \lim_{h \to 0} \left( \sin(b) \frac{(\cos(h) - 1)}{h} + \cos(b) \frac{\sin(h)}{h} \right) \\ &= \sin(b) \lim_{h \to 0} \frac{(\cos(h) - 1)}{h} + \cos(b) \lim_{h \to 0} \frac{\sin(h)}{h} \\ &= \sin(b)\, 0 + \cos(b)\, 1 \\ &= \cos(b) \end{aligned}$$

The tangent slope to the graph of sine at the point $(b, \sin(b))$ is $\cos(b)$.

### 15.2 Tangent slope to the graph of cosine at the point $(b, \cos(b))$.

The tangent slope at the graph point $(b, \cos(b))$ is the limit of the difference quotient:

$$\frac{\cos(\,b+h\,)\;-\;\cos(\,b\,)}{h}\,.$$

We use the formula for the cosine of a sum to get:

$$\begin{aligned}\cos(\,b+h\,)\;-\;\cos(\,b\,) &= \cos(b)\,\cos(h)\;-\;\sin(b)\,\sin(h)\;-\;\cos(b)\\ &= \cos(b)\,(\,\cos(h)\;-\;1\,)\;-\;\sin(b)\,\sin(h)\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(\,b+h\,)\;-\;\cos(\,b\,)}{h} &= \lim_{h\to 0}\left(\cos(b)\,\frac{(\,\cos(h)\;-\;1\,)}{h}\;-\;\sin(b)\,\frac{\sin(h)}{h}\right)\\ &= \cos(b)\;\lim_{h\to 0}\frac{(\,\cos(h)\;-\;1\,)}{h}\;-\;\sin(b)\;\lim_{h\to 0}\frac{\sin(h)}{h}\end{aligned}$$

Our earlier calculation using the squeeze theorem showed

$$\lim_{h\to 0}\frac{(\,\cos(h)\;-\;1\,)}{h}\;=\;0\quad\text{and}\quad\lim_{h\to 0}\frac{\sin(h)}{h}\;=\;1\,;$$

so,

$$\lim_{h\to 0}\frac{\cos(\,b+h\,)\;-\;\cos(\,b\,)}{h}\;=\;\cos(b)\;0\;-\;\sin(b)\;1\;=\;-\sin(b)$$

The tangent slope to the graph of cosine at the point $(b, \cos(b))$ is $-\sin(b)$.

## 16 More derivative calculations

Examples:

Using the two important limits

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{1 - \cos(h)}{h} = 0 ,$$

we earlier calculated:

- $$\begin{aligned} \sin(\,b+h\,) - \sin(\,b\,) &= \sin(b)\cos(h) + \cos(b)\sin(h) - \sin(b) \\ &= \sin(b)\,(\,\cos(h) - 1\,) + \cos(b)\sin(h) \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0} \frac{\sin(\,b+h\,) - \sin(\,b\,)}{h} &= \lim_{h\to 0} \left( \sin(b)\,\frac{(\,\cos(h) - 1\,)}{h} + \cos(b)\,\frac{\sin(h)}{h} \right) \\ &= \sin(b) \lim_{h\to 0} \frac{(\,\cos(h) - 1\,)}{h} + \cos(b) \lim_{h\to 0} \frac{\sin(h)}{h} \\ &= \sin(b)\ 0 + \cos(b)\ 1 \\ &= \cos(b) \end{aligned}$$

The derivative (tangent slope) of the sine function is the cosine function.

- $$\begin{aligned} \cos(\,b+h\,) - \cos(\,b\,) &= \cos(b)\cos(h) - \sin(b)\sin(h) - \cos(b) \\ &= \cos(b)\,(\,\cos(h) - 1\,) - \sin(b)\sin(h) \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0} \frac{\cos(\,b+h\,) - \cos(\,b\,)}{h} &= \lim_{h\to 0} \left( \cos(b)\,\frac{(\,\cos(h) - 1\,)}{h} - \sin(b)\,\frac{\sin(h)}{h} \right) \\ &= \cos(b) \lim_{h\to 0} \frac{(\,\cos(h) - 1\,)}{h} - \sin(b) \lim_{h\to 0} \frac{\sin(h)}{h} \\ &= \cos(b)\ 0 - \sin(b)\ 1 \\ &= -\sin(b) \end{aligned}$$

So, the derivative of the cosine function is minus the sine function.