Applications of derivatives: maximum and minimum

Some definitions

Suppose f is a function with domain an interval $\mathcal{I}$ (which may or may not include the end-points).

- A input c provides an **absolute maximum** of f, if

$$f(c) \geq f(y) \quad \text{for all } y \text{ in the interval } \mathcal{I}$$

  The value $f(c)$ is called the **absolute maximum value of** f.

- A input c provides an **absolute minimum** of f, if

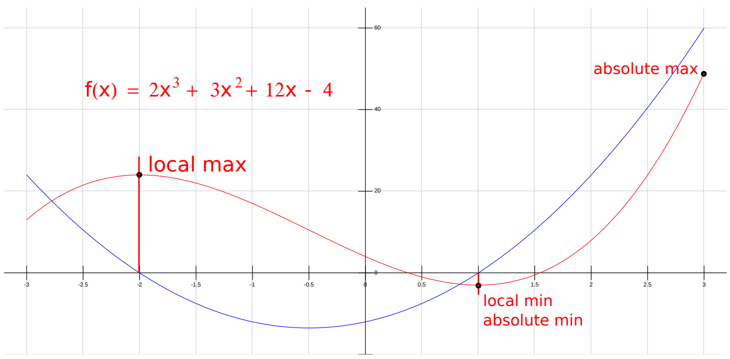
$$f(c) \leq f(y) \quad \text{for all } y \text{ in the interval } \mathcal{I}$$

  The value $f(c)$ is called the **absolute minimum value of** f.

- Absolute maximum/minimum are also often called **global maximum/minimum** as well as **extreme values**.
- An **interior point** c is a point in $\mathcal{I}$ so that there are points of $\mathcal{I}$ both left and right of c. This is the same as saying c belongs to $\mathcal{I}$, but is not an endpoint.
- An interior point c provides a **local maximum** if there is a 'small' interval $\mathcal{J}$ around c so that c provides an absolute maximum on $\mathcal{J}$. Similarly for **local minimum**.

Example. We take the function $f(x) = 2x^3 + 3x^2 - 12x + 4$ on the interval $\mathcal{I} = [-3, 3]$.

x	$f(x)$	$f'(x)$
3	49	
2	8	
1	-3	0
0	4	-12
-1	14	-12
-2	24	0
-3	13	



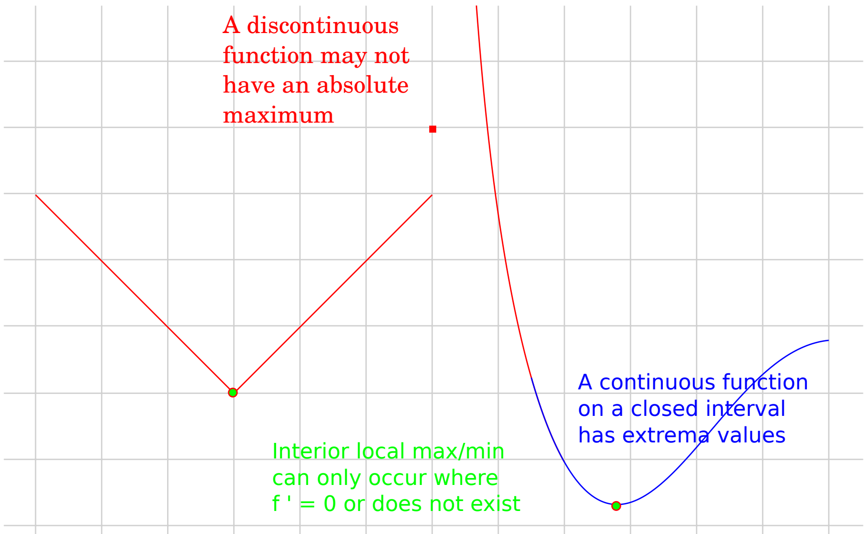
Two Theorems about extreme values

Extreme Value Theorem. Suppose f is a continuous function on a closed interval $\mathcal{I} = [a, b]$. Then f will take an absolute maximum value at some point in $\mathcal{I}$. It will also take on an absolute minimum value.

If we have a continuous function f on a closed interval, the Extreme Value Theorem tells us the function will have absolute max/min values. In our search for these values, we know they are there to be found.

Local Extreme Value Theorem (Fermat's Theorem). Suppose f is a function on interval $\mathcal{I}$ (not necessarily closed). If f has a local max/min at the interior point c of $\mathcal{I}$, and $f'(c)$ exists, then $f'(c) = 0$.

The usefulness of the local extreme value theorem is it helps us to locate interior points which provide local max/min of a function f. If the function f is differentiable, we will find local max/min among the inputs c where $f'(c) = 0$.



- If a function f is defined on an interval $\mathcal{I}$, an interior point c is called a **critical point** if either:

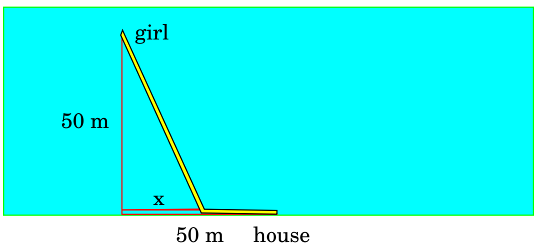
$$\text{the derivative does not exists at } c, \quad \text{or} \quad f'(c) = 0.$$

Inputs where absolute and local max/min can occur.

Suppose f is a continuous function on a closed interval $\mathcal{I} = [a, b]$. Then, the possible inputs where the absolute and local max/min can occur are:

- Critical points in the interior. Reminder. A critical point is an input in the interior where either the derivative does not exists or exists and is zero.
- The endpoints.

Example A girl in the ocean is 50 meters from shore. She wishes to travel to a house 50 meters along the shore from the closest point to her ocean location.



She can swim at a rate of 2 m/sec and walk at a rate of 4 m/sec. Locate the point x on shore which will minimize the travel time. We have:

$$\text{swim time} = \frac{\sqrt{50^2 + x^2}}{2}, \quad \text{walk time} = \frac{50 - x}{4}$$

$$\text{total time } T(x) = \frac{\sqrt{50^2 + x^2}}{2} + \frac{50 - x}{4}$$

The domain of F is the closed interval $[0, 50]$, and we seek an absolute minimum.

- Since T is continuous on $[0, 50]$, the extreme value theorem says T will have absolute max/min values.
- T is differentiable. The combnination of the extreme value theorem and the local extreme value theorem tells us the **absolute max/min** must occur at either an interior point where T' is zero or an endpoint.

We use the rules to find the derivative:

$$T'(x) \;=\; \frac{1}{2}\Big(\frac{1}{2}\cdot(50^2+x^2)^{-\frac{1}{2}}\cdot 2\,x\Big) \;-\; \frac{1}{4} \;=\; \frac{x}{2\sqrt{50^2+x^2}} \;-\; \frac{1}{4}$$

The condition $T' = 0$ becomes:

$$0 \;=\; \frac{x}{\sqrt{50^2+x^2}} \;-\; \frac{1}{2}\,, \quad\Longrightarrow\quad 4x^2 \;=\; 50^2 \;+\; x^2, \quad\Longrightarrow\quad 3x^2 \;=\; 50^2 \;\text{ so }\; x \;=\; \pm\frac{50}{\sqrt{3}}$$

The interior critical point is therefore $x = \frac{50}{\sqrt{3}}$. The absolute minimum will occur either at $x = \frac{50}{\sqrt{3}}$ or the endpoints 0 and 50. We make a table of values

x	$f(x)$
0	$\frac{50}{2} + \frac{50}{4} = 37.5$
50	$\frac{50\sqrt{2}}{2} = 35.3555$
$\frac{50}{\sqrt{3}}$	$\frac{25\sqrt{3}}{2} + \frac{25}{2} = 34.1506$

We see the absolute minimum (shortest time) is provided by the (interior) critical point $x = \frac{50}{\sqrt{3}}$.