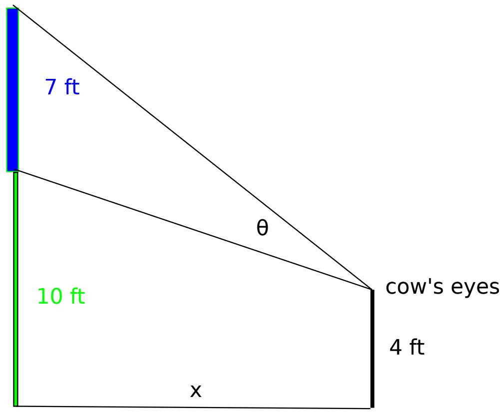
Applications of derivatives to optimization problems.

Derivatives were discovered because they are so useful in determining the local max/min of common functions. They are still used for this purpose.

Example 1. Webwork problem on near-sighted cow.

A 'near-sighted cow' whose eye level is 4 ft wishes to watch a movie screen which is 7 ft high and whose bottom is 10 ft above ground level.



Determine the angle θ as a function of x, then find the absolute maximum value of θ.

Angle θ is a difference $\theta = \theta_{13} - \theta_6$, where

$$\tan(\theta_{13}) = \frac{13}{x} \quad \text{and} \tan(\theta_{13}) = \frac{6}{x}.$$

So,

$$\theta(x) = \arctan(\frac{13}{x}) - \arctan(\frac{6}{x}) \quad \text{with domain } 0 < x.$$

- Limit as $x \to 0^+$. We note that both

$$\lim_{x\to 0^+} \arctan(\frac{13}{x}) = \frac{\pi}{2}, \quad \text{and} \quad \lim_{x\to 0^+} \arctan(\frac{6}{x}) = \frac{\pi}{2}, \quad \text{so} \quad \lim_{x\to 0^+} \theta(x) = 0.$$

  This means if we set $\theta(0) = 0$, we have a continuous function on $[0, \infty)$.

- Limit as $x \to \infty$. As $x \to \infty$, both $\frac{13}{x}$ and $\frac{6}{x}$ have limit 0 (from above), so

$$\lim_{x\to\infty} \theta(x) = 0$$

  The x-axis is a horizontal asymptote.

- The function θ, as difference of two arctan functions is differential on the interval $(0, \infty)$, so critical point are where the derivative is zero. The derivative is:

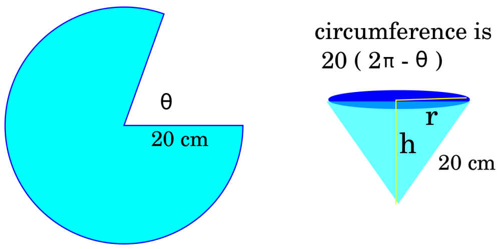
$$\theta'(x) = \big(\arctan(\frac{13}{x}) - \arctan(\frac{6}{x})\big)' = \frac{1}{1+\left(\frac{13}{x}\right)^2}\,(-\frac{13}{x^2}) - \frac{1}{1+\left(\frac{6}{x}\right)^2}\,(-\frac{6}{x^2})$$
$$= \frac{-13}{x^2+13^2} + \frac{6}{x^2+6^2} = \frac{-13\,(x^2+6^2) + 6\,(x^2+13^2)}{(x^2+13^2)(x^2+6^2)}$$
$$= \frac{-7\,x^2+13\cdot 6\cdot 7}{(x^2+13^2)(x^2+6^2)} = 7\,\frac{-x^2+13\cdot 6}{(x^2+13^2)(x^2+6^2)}$$

The derivative has the same sign as $78-x^2$. In the interval $(0,\infty)$, there is a single critcal point at $c=\sqrt{78}=8.63\ldots$, and clearly the inverted paraboloa shape of $78-x^2$ means the derivative will flip signs from $+$ to $-$, so c is a local maximum. Since the limits as x approaches 0 and ∞ are 0, the local maximum at $c=\sqrt{78}=8.63\ldots$ feet is an absolute maximumn. The value is

$$\theta(\sqrt{78}) = \arctan(\frac{13}{\sqrt{78}}) - \arctan(\frac{6}{\sqrt{78}}) = 0.3773\ldots \quad \text{radians (21.618 degrees)}$$

Example 2. Creation of cone with maximum volume.

A sector is removed from a disk of paper of radius 20 cm, and the part left is made into a cone.



Determine the function for the volume V of the cone as a function of θ, and then find the absolute maximum volume.

Solution.

The circumference of the (top) rim of the cone is $20\,(2\pi - \theta)$. The top radius r satisfies:

$$2\,\pi\,r = (2\pi - \theta)\,, \quad \text{so} \quad r = \frac{(2\pi - \theta)}{2\,\pi} = 20\left(1 - \frac{\theta}{2\pi}\right).$$

The relationship between the height h and r is:

$$h^2 + r^2 = 20^2\,, \quad \text{so} \quad h = \sqrt{20^2 - r^2} = 20\sqrt{1-\left(1-\frac{\theta}{2\pi}\right)^2}\,.$$

The volume is:

$$\begin{aligned} V(\theta) &= \frac{1}{3} \cdot \text{area of base} \cdot \text{height} \\ &= \frac{1}{3} \cdot \pi r^2 \cdot h \\ &= \frac{1}{3} \cdot \pi \left(20\left(1 - \frac{\theta}{2\pi}\right)\right)^2 \cdot 20\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2} \\ &= \frac{20^3\,\pi}{3} \cdot \pi\left(1 - \frac{\theta}{2\pi}\right)^2 \cdot \sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2} \end{aligned}$$

Two tricks:

- Set $x = \left(1 - \frac{\theta}{2\pi}\right)$. Then,

$$V = \frac{20^3\,\pi}{3} x^2 \sqrt{1 - x^2}\ ,$$

which is simpler than the expression in θ. As a function of θ or of x the absolute maximum is the same. The domain for the input θ is $[0, 2\pi]$, so the domain of the input x is $[0, 1]$.

- The square V^2, has the simpler expression:

$$V^2 = \left(\frac{20^3\,\pi}{3}\right)^2 x^4 \left(1 - x^2\right)$$

The input x that maximizes V is the same as the input x that maximizes $F = V^2$. So it is enough to maximize $F(x) = x^4 \cdot (1 - x^2)$ on the interval $[0, 1]$

The derivative $\frac{dF}{dx}$ of F is:

$$\begin{aligned} F'(x) &= \left(x^4 \cdot (1 - x^2)\right)' \\ &= 4x^3(1 - x^2) + x^4(0 - 2x) \\ &= x^3\left(4(1 - x^2) - 2x^2\right) = x^3\left(4 - x^2 - 2x^2\right) \\ &= x^3\, 2\,(2 - 3x^2)\ . \end{aligned}$$

We see

- right-sided derivative of F at 0 equals zero
- $c = \sqrt{\frac{2}{3}}$ is interior critical point.

$F' > 0$ on interval $0 < x < c \implies F$ increasing from $F(0) = 0$ to $F(c)$

$F' < 0$ on interval $c < x < 1 \implies F$ decreasing from $F(c)$ to $F(1) = 0$

So input c provides a local maximum and in fact absolute maximum.

We have

$$\sqrt{\frac{2}{3}} = x\left(1 - \frac{\theta}{2\pi}\right) \text{ so } \theta = 2\pi\left(1 - \sqrt{\frac{2}{3}}\right) = 1.1529\ldots \text{ radians (66.06123 degrees)}$$

$$\begin{aligned} V\Big|_{x=\sqrt{\frac{2}{3}}} &= \frac{20^3\,\pi}{3}\left(\sqrt{\frac{2}{3}}\right)^2 \sqrt{\left(1 - \left(\sqrt{\frac{2}{3}}\right)^2\right)} \\ &= \frac{20^3\,\pi}{3}\,\frac{2}{3}\,\frac{1}{\sqrt{3}} = 783,561.3\ldots\ \text{cm}^3 \end{aligned}$$

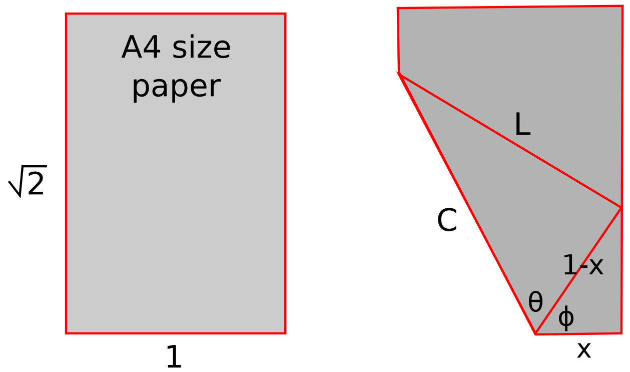
Example 3. Creation of fold of A4 paper with extreme length.

Paper sizes of A1, A2, A3, A4, are based on mathematics. The ratio of the height to width is $\sqrt{2}$ to 1.

A3-size paper is folded in half along height gives A4-size paper.

A4-size paper is folded in half along height gives A5-size paper.

In figure consider folding the bottom left corner so that it lies on the right edge. Determine the function C which is the crease length in terms of an appropriate variable (x, or ϕ, or θ).



We see

$$\cos(\theta) = \frac{1-x}{C}\ , \quad \phi + \theta + \theta = \pi \text{ radians } (180^\circ)\ , \quad \cos(\phi) = \frac{x}{1-x}\ .$$

So, $\theta = (\frac{\pi}{2} - \frac{\phi}{2})$, and

$$C = \frac{1-x}{\cos(\theta)} = \frac{1-x}{\cos(\frac{\pi}{2} - \frac{\phi}{2})} = \frac{1-x}{\sin(\frac{\phi}{2})}\ .$$

To determine $\sin(\frac{\phi}{2})$, we use the trig-identity $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ with $A = B = \frac{\phi}{2}$ to get:

$$\begin{aligned} \cos(\phi) &= \cos(\frac{\phi}{2} + \frac{\phi}{2}) = \cos(\frac{\phi}{2})\cos(\frac{\phi}{2}) - \sin(\frac{\phi}{2})\sin(\frac{\phi}{2}) \\ &= \left(\cos(\frac{\phi}{2})\right)^2 - \left(\sin(\frac{\phi}{2})\right)^2\ , \quad \text{now add and substract } \left(\sin(\frac{\phi}{2})\right)^2 \\ &= 1 - 2\,(\sin(\frac{\phi}{2}))^2\ . \end{aligned}$$

So, solving for $(\sin(\frac{\phi}{2}))^2$ gives:

$$\begin{aligned} (\sin(\frac{\phi}{2}))^2 &= \frac{1-\cos(\phi)}{2} = \frac{1-\frac{x}{1-x}}{2} = \frac{1-2x}{2(1-x)} \\ \sin(\frac{\phi}{2}) &= \sqrt{\frac{1-2x}{2(1-x)}} \quad \text{and so} \\ C &= \frac{1-x}{\sin(\frac{\phi}{2})} = \frac{1-x}{\sqrt{\frac{1-2x}{2(1-x)}}} = \frac{\sqrt{2}\,(1-x)^{\frac{3}{2}}}{1-2x} \end{aligned}$$

The algebraic function/rule of x on the RHS has domain $[0, \frac{1}{2}]$, but physically, this is not the domain of C. To determine the actual domain, we note

$$\lim_{x \to \frac{1}{2}} \frac{\sqrt{2}\,(1-x)^{\frac{3}{2}}}{1-2x} \;=\; \infty\ .$$

The height $\sqrt{2}$ imposes a condition on the input x:

$$\begin{aligned} \frac{L}{1-x} \;&=\; \tan(\theta) \quad \text{and} \quad L \;\leq\; \sqrt{2} \;\; \text{(the vertical length), so} \\ (1\;-\;x)\;\tan(\theta) \;&=\; L \;\leq \sqrt{2} \end{aligned}$$

Since $\theta = \frac{\pi}{2} - \frac{\phi}{2}$,

$$\begin{aligned} \tan(\theta) \;&=\; \tan(\frac{\pi}{2} - \frac{\phi}{2}) \;=\; \frac{\cos(\frac{\phi}{2})}{\sin(\frac{\phi}{2})} \\ (\,\tan(\theta)\,)^2 \;&=\; \frac{\big(\,\cos(\frac{\phi}{2})\,\big)^2}{\big(\,\sin(\frac{\phi}{2})\,\big)^2} \;=\; \frac{1\;-\;\big(\,\sin(\frac{\phi}{2})\,\big)^2}{\big(\,\sin(\frac{\phi}{2})\,\big)^2} \;=\; \frac{1\;-\;\frac{1-2x}{2(1-x)}}{\frac{1-2x}{2(1-x)}} \;=\; \frac{1}{1-2x} \end{aligned}$$

The condition $(1\;-\;x)\;\tan(\theta) \;=\; L \;\leq \sqrt{2}$ becomes:

$$(1-x)^2\,\frac{1}{1-2x} \;\leq\; 2\;, \;\; \text{so} \;\; 1\;-\;2x\;+x^2 \;\leq\; 2\;-\;4x\;, \;\; \text{so} \;\; (1\;+\;x)^2 \;\leq 2\;.$$

This limits the input x to $x \leq \sqrt{2}-1$. The A4 height $\sqrt{2}$ means the domain of C is $[0, (\sqrt{2}-1)]$.

The function C is continuous on $[0, (\sqrt{2}-1)]$, and differentiable in the interior. The global and local extreme value theorem tell us the absolute/local max/min will occur at either the endpoint $x = 0, (\sqrt{2}-1)$ or at interior critical points.

Since C has a square-root, we use the trick that the inputs which make C has absolute/local max/min are the same as for the square:

$$C^2 \;=\; 2\,\frac{(1-x)^3}{1-2x} \qquad (\,\text{so} \quad C^2(0) \;=\; 2, \text{ and } C^2(\sqrt{2}-1) \;=\; 2.34314\ldots\,)\,.$$

To find the interior critical points:

$$\begin{aligned} (\,C^2\,)'' \;&=\; 2\Big(\frac{(1-x)^3}{1-2x}\Big)' \;=\; 2\,\frac{3\,(1-x)^2\,(-1)\,(1-2x)\;-\;(1-x)^3\,(\,0\;-\;2\,)}{(\,1-2x\,)^2} \\ &= 2\,(1-x)^2\,\frac{3\,(-1)\,(1-2x)\;-\;(1-x)\,(-2)}{(\,1-2x\,)^2} \\ &=\; 2\,\frac{(1-x)^2}{(1-2x)^2}\,(\,6x\;-\;3\;+\;2\;-\;2x\,) \;=\; 2\,\frac{(1-x)^2}{(1-2x)^2}\,(\,4x\;-\;1\,) \end{aligned}$$

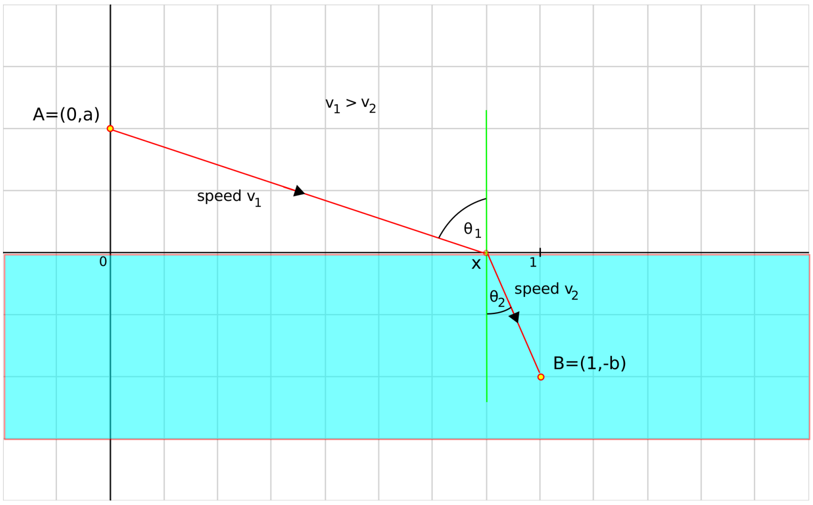
We see $(C^2)' < 0$ in the interval $(0, \frac{1}{4})$ and then $(C^2)' > 0$ afterwards. We deduce C^2 has:

- an absolute minimum at $x = \frac{1}{4}$,
- an absolute maximum at $x = (\sqrt{2}-1)$.

To create a crease of minimum length, we fold at $x = \frac{1}{4}$.

Example 4. Snell's Law on shortest time path.

The speed of light is faster in air than glass (or water). The ratio of the two speeds $\dfrac{v_{\text{air}}}{v_{\text{glass}}}$ is called the air/glass index of refraction and it is approximately $\frac{3}{2}$ (the refraction index is dependent on particular type of glass and the color (wavelength) of the light). Similarly, there is an air/water index of refraction $\dfrac{v_{\text{air}}}{v_{\text{water}}}$ whose value is approximately $\frac{4}{3}$.



We determine the shortest time path to travel from point A to point B. We choose the coordinate system so $A = (0, a)$, and $B = (1, -b)$.

The total travel time from A to B is:

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (1-x)^2}}{v_2} .$$

The domain of T is the interval $[0, 1]$. F is continuous on $[0, 1]$ and differentiable in the interior. By the global/local extreme value theorems, the absoluet mimimum exists and occurs at either the endpoints $x = 0, 1$ or at in interior critical point. The derivative of T is:

$$T'(x) = \left(\frac{1}{v_1}\right)\frac{1}{2}\,(a^2 + x^2)^{-\frac{1}{2}}\,(2\,x) + \left(\frac{1}{v_2}\right)\frac{1}{2}\,(b^2 + (1-x)^2)^{-\frac{1}{2}}\,(2\,(1-x)\,(-1))$$

$$= \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{(1-x)}{v_2\sqrt{b^2 + (1-x)^2}}$$

$$= \frac{\sin(\theta_1)}{v_1} - \frac{\sin(\theta_2)}{v_2}$$

Note. $T'(0) = -\dfrac{\sin(\,\theta_{2|_{x=0}}\,)}{v_2} < 0$, and $T'(1) = \dfrac{\sin(\,\theta_{1|_{x=1}}\,)}{v_1} > 0$. As x moves from 0 to 1, $\sin(\theta_1)$ increases and $\sin(\theta_2)$ decreases; therefore T' is increasing on $(0, 1)$. There will be exactly one critical point c where T' flips from $-$ to $+$. It gives an absolute minimum, and at the point c, we have

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}\ ,\ \text{so}\ \frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2} .$$

This is called called Snell's Law.