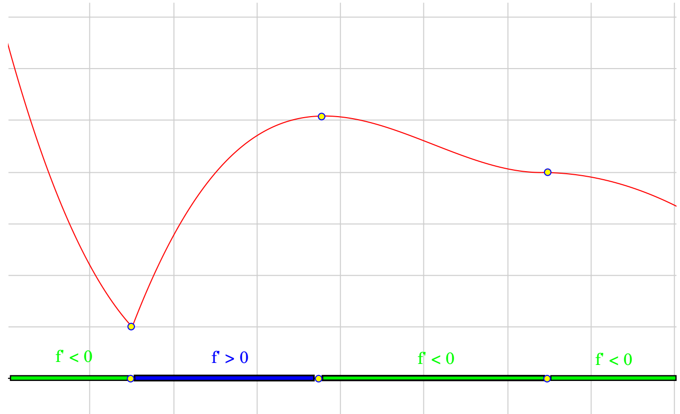
1st derivative test.

When a function is continuous on a closed interval $[a, b]$, and c is an interior critical point, one tries to find an interval $[x_1, x_2]$ around c so that f' has the same sign on (x_1, c) and the opposite sign on (c, x_2). Then c must provide an absolute maximum or minimum on the interval $[x_1, x_2]$, and so c provides a local maximum or minimum (on the entire interval $[a, b]$).

- If f' changes sign from $+$ to $-$ (or from $-$ to $+$) as the input increases through an interior critical point c, then c provides a local maximum (or a local minimum).



- If f' does not change sign as the input increases through an interior critical point c, then c provides a neither a local max/min.

A simple method/algorithm to determine if the derivative f' of a function f flips sign as the input passes through an critical point is:

2nd derivative test.

Suppose $f'(c) = 0$ is a critical point of a function f, and that the 2nd derivative f'' exists in an interval $[x_1, x_2]$ surrounding c.

- If $f'' > 0$, then f' is increasing on $[x_1, x_2]$, and so it must change signs from $-$ to $+$ as the input increases through critical point c (where $f'(c) = 0$). So the critical point c is a local minimum.
- If $f'' < 0$, then f' is decreasing on $[x_1, x_2]$, and so it must change signs from $+$ to $-$ as the input increases through critical point c (where $f'(c) = 0$). So the critical point c is a local minimum.

In particular, if f'' is continuous in an interval surrounding the critcal point c, then

$$\begin{aligned} f''(c) &> 0 \quad \Longrightarrow \quad c \text{ provides a local minimum} \\ f''(c) &< 0 \quad \Longrightarrow \quad c \text{ provides a local maximum} \end{aligned}$$

Examples

- Redo the example
$$f(x) \;=\; e^{x^3-x}$$
and use the 2nd derivative test to determine what happens at the critical points $\pm\frac{1}{\sqrt{3}}$. We have
$$\begin{aligned} f'(x) &= e^{x^3-x}\,(\,3x^2 \;-\; 1\,) \\ f''(x) &= e^{x^3-x}\,(\,3x^2 \;-\; 1\,)\,(\,3x^2 \;-\; 1\,) \;+\; e^{x^3-x}\,(\,6x\,) \\ &= e^{x^3-x}\,\big(\,(\,3x^2 \;-\; 1\,)^2 \;+\; 6x\,\big) \end{aligned}$$
So,
$$\begin{aligned} f''(\frac{1}{\sqrt{3}}) &= e^{\cdots}\,\big(\,0^3 \;+\; 6\frac{1}{\sqrt{3}}\,\big) \;>0 \quad \text{local minimum} \\ f''(-\frac{1}{\sqrt{3}}) &= e^{\cdots}\,\big(\,0^3 \;+\; 6\,(-\frac{1}{\sqrt{3}})\,\big) \;<0 \quad \text{local maximum} \end{aligned}$$
Note. $e^{\cdots}$ is positive, so we can disregard it since we are only interested in the sign.

- Find critical points of $f(x) \;=\; x\,e^{-2x^2}$ on the interval $(-\infty, \infty)$ and use the 2nd derivative test to determine local max/min. Note the function is an odd function. We have
$$\begin{aligned} f'(x) &= 1\,e^{-2x^2} \;+\; x\,e^{-2x^2}\,(-4x) \\ &= e^{-2x^2}\,(\,1 \;-\; 4x^2\,) \qquad \text{so critical points at } \pm\frac{1}{2} \\ f''(x) &= e^{-2x^2}\,(-4x)\,(\,1 \;-\; 4x^2\,) \;+\; e^{-2x^2}\,(\,0 \;-\; 8x\,) \\ &= e^{-2x^2}\,(\,16x^3 - 12x\,) \;=\; 4\,e^{-2x^2}\,x\,(\,4x^2 \;-\; 3\,) \end{aligned}$$
Then,
$$\begin{aligned} f''(\frac{1}{2}) &= 4\,e^{\cdots}\,(\frac{1}{2})\,(\,4\,(\frac{1}{2})^2 \;-\; 3\,) < 0 \quad \text{local maximum} \\ f''(-\frac{1}{2}) &= 4\,e^{\cdots}\,(-\frac{1}{2})\,(\,4\,(-\frac{1}{2})^2 \;-\; 3\,) > 0 \quad \text{local minimum} \end{aligned}$$
Note that we could have also use factor $(1 - 4x^2)$ in the derivative to also determine how it flips sign at the two critical points.

Caution. In case $f''(c) \;=\; 0$, there is no statement. Different examples exists where $f''(c) = 0$, in which the input c is a local maximum, local minimum, or neither.

The 2nd dereivative test can be viewed as a ‘no brain thinking’ test. Often one can inspect the derivative directly to determine if it flips sign as the input increases through a critical point.

The 2nd derivative and the shape of the graph.

Besides it role in determining whether a critical point is a local max/min, the sign of the 2nd derivative f'' is very important in understanding the shape of the graph of f. The key fact/intuition is again:

$$(f')' \text{ positive/negative} \quad \Longrightarrow \quad (f') \text{ is increasing/decreasing.}$$

A thought experiment.

Suppose we have two cars A, and B moving on an axis. Let the functions $A(t)$, $B(t)$ be the positions. Suppose:

- At time $t = 0$, the two cars are at the same position, and the velocity of car A is at least that of car B; so
$$A(0) \;=\; B(0) \quad \text{and} \quad A'(0) \;\geq\; B'(0)\ .$$
- For $t > 0$, the acceleration of A is greater than that of B, so
$$A''(t) \;=\; B''(t)$$

How do the positions $A(t)$, $B(t)$ and velocities $A'(t)$, $B'(t)$ compare?
Set $g(t) = A(t) - B(t)$, the difference/gap in positions. The hypotheses are then

$$g(0) \;=\; 0 \;\;,\quad g'(0) \;>\; 0 \;\;,\quad g''(t) \;>\; 0 \;\; \text{all } t \;>\; 0\ .$$

Base on our everyday common sense, we expect:

- For $t > 0$, the difference in velocities $g'(t) = A'(t) - B'(t)$ is increasing since car A has greater acceleration than car B.
- For $t > 0$, the difference in positions $g(t) = A(t) - B(t)$, which is zero at $t = 0$ should be increasing as car A has greater velocity. In fact, the gap $A(t) - B(t)$ should be widening.

We now explain how the 2nd derivative f'' shapes the graph of f. Suppose $f'' > 0$ everywhere in the domain $[a, b]$. We take f as our function A (position of car A). For the function B (position of car B), we take a tangent line. For example the tangent line at interior point point d. So

$$B_d(t) \ = \ f'(d)\,(\,t \ - \ d\,) \ + \ f(d)\ .$$

Let $g(t) \ = \ f(t) \ - \ B_d(t)$ be the gap function. Then clearly

$$g(d) \ = 0 \quad , \quad g'(d) \ = \ 0 \quad , \quad g''(t) = f''(t) \ - \ B_d''(t) \ = \ f''(t) \ > \ 0 \ \text{ all } t \ > \ d$$

satisfy the hypotheses of the thought experiment.

Conclusion 1. For $t > d$, the graph of f is **above** its tangent line B_d at d, and the gap $g(t) \ = \ f(t) \ - \ B_d(t)$ widens for increasing t.

What about $t < d$? The function

$$G(t) \ = \ g(\,2d \ - \ t\,)$$

is the gap function g reflected across the vertical line $t = d$. Input $t > d$ to G corresponds to input $2d - t$ to g. We have

$$\begin{aligned} G(d) \ &= \ g(d) \ = \ 0 \\ G'(t) \ &= \ g'(\,2d \ - \ t\,)\,(-1) \ \text{ so } \ G'(d) \ = \ g'(d) \ = \ 0 \\ G''(t) \ &= \ g''(\,2d \ - \ t\,)\,(-1)\,(-1) \ = f''(\,2d \ - \ t\,) \ > \ 0 \end{aligned}$$

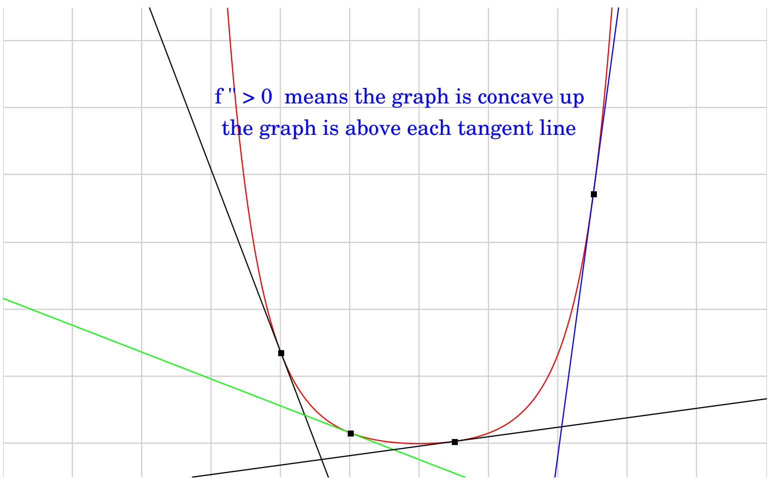
Conclusion 2. For $t > d$, the graph of f is also **above** its tangent line B_d at d. And, the gap $g(t) \ = \ f(t) \ - \ B_d(t)$ widens as t moves away from d.

2nd derivative f'' and the shape of the graph of f.

Suppose f is a continuous function with domain an interval $[a, b]$, and:

- Both the 1st and 2nd derivatives f' and f'' exists at all points in the interior.
- $f'' > 0$ in the interior.

Then, the graph of f is always above any of its tangent lines, and as the input moves away from where the tangent line meets the graph, the gap widens. We say the shape of the graph is **concave up**.



Similarly, if $f'' < 0$, the graph of f will be below each of its tangent lines and as the input moves away from where the tangent line meets the graph, the gap widens. We say the shape of the graph is **concave down**.

Subintervals of concave up and down, and graphing f.

Given a function f with domain an interval, and whose derivatives f' and f'' exists in the interior, one tries to divide the interval up into subintervals where f'' has the same sign at all inputs of the subinterval. On each subinterval the graph of f will either be concave up ($f'' > 0$) or concave down ($f'' < 0$). This information is very useful for sketching the graph of the function.

A point where the concavity switches from up to down or vice versa is called an **inflection point**.

Example. Analyze the function $f(x) = x^2 e^{-x}$ on the interval $(-\infty, \infty)$ for:

- critical points
- local max/min
- concavity intervals and inflection points

The derivative f' exists so critical points will be where $f' = 0$. We have

$$f'(x) = 2x\, e^{-x} + x^2 e^{-x}(-1) = e^{-x}(2x - x^2)\,.$$

Therefore, the critical point are where $(2x - x^2) = x(2-x)$ is zero. They are $x = 0$, or 2. The sign of the derivative is the same as the sign of $x(2 - x)$. We see from this the sign of f' switches:

- from $-$ to $+$ as x increases through 0 so the critical point 0 is a local minimum.
- from $+$ to $-$ as x increases through 2 so the critical point 2 is a local maximum.

We have $f(0) = 0$ and $f(2) = 4e^{-2}$.
To compute concavity subintervals, we compute:

$$\begin{aligned} f''(x) &= \left(e^{-x}(2x - x^2)\right)' = e^{-x}(-1)(2x - x^2) + e^{-x}(2 - 2x) \\ &= e^{-x}(x^2 - 2x + 2 - 2x) = e^{-x}(x^2 - 4x + 2)\,. \end{aligned}$$

So f'' has the same sign as $(x^2 - 4x + 2)$, which has roots:

$$\frac{-(-4) \pm \sqrt{16 - 8}}{2} = 2 - \sqrt{2} \text{ and } 2 + \sqrt{2}\,.$$

We see

$$f'' > 0 \text{ for } x < (2 - \sqrt{2}), \text{ so the graph is concave up}$$
$$f'' < 0 \text{ for } (2 - \sqrt{2}) < x < (2 + \sqrt{2}), \text{ so the graph is concave down}$$
$$f'' > 0 \text{ for } (2 + \sqrt{2}) < x, \text{ so the graph is concave up}$$

The inflection points are at $(2 - \sqrt{2})$, and $(2 + \sqrt{2})$.

x	$f(x)$
0	0
2	0.5413
$2-\sqrt{2}$	0.1210
$2+\sqrt{2}$	0.3835