Integration.

The word **integration** means to assemble. In calculus, the word integration refers to a way of calculating areas or volumes of objects by taking the limit of certain sums. The ability to find limits of sums then has many applications besides just finding areas and volumes.

Sigma notation.

If we have a list of numbers a_1, a_2, $\ldots$, a_m, and we wish to add all the numbers together, we use the notation:

$$\sum_{k=1}^{m} a_k \quad \text{means the sum} \quad (a_1 + \cdots + a_m)$$

Examples.

- $\sum_{k=0}^{8} (4 + 5k)$ means to sum the 9 values $f(0)$, $f(1)$, $\ldots$, $f(8)$ of the function $f(k) = (4 + 5k)$. We have $f(0) = 4$, and $f(8) = 44$; so,

$$\sum_{k=0}^{8} (4 + 5k) = f(0) + f(1) + \cdots + f(8) = 4 + 9 + 14 + \cdots + 44$$

- $\sum_{\ell=1}^{49} \frac{1}{\ell(\ell+1)}$ means to sum the 49 values $g(1)$, $\ldots$, $g(49)$ of the function $g(\ell) = \frac{1}{\ell(\ell+1)}$.
so,

$$\sum_{\ell=1}^{49} \frac{1}{\ell(\ell+1)} = g(1) + g(2) + \cdots + g(49) = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{49 \cdot 50}$$

Intuitive properties of area:

- A rectangle of base b and height h has area $b\,h$.
- If a region X can be partitioned into 2 non-overlapping regions A and B, then

$$\text{area}(X) \;=\; \text{area}(A) \;+\; \text{area}(B)\ .$$

  More generally, if a region X can be partitioned into non-overlapping regions $X_1,\ X_2,\ \ldots\ ,\ X_n$, then

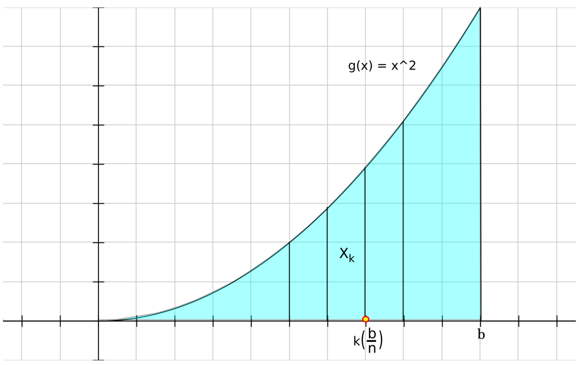
$$\text{area}(X) \;=\; \sum_{k=1}^{n} \text{area}(X_k)$$

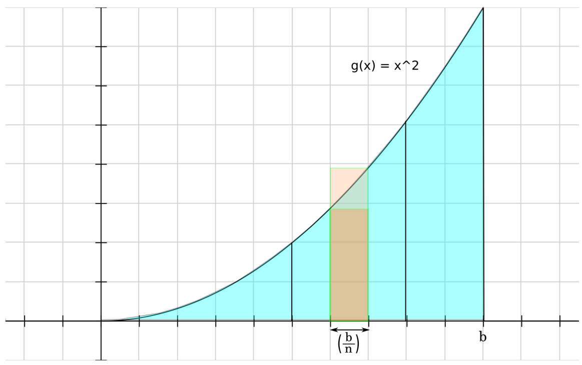
- If A and B are 2 regions and A is contained in B, then

$$\text{area}(A) \;\leq\; \text{area}(B)$$

The above three properties can be combined with limits to determine some areas of ‘complicated’ regions.

Example. Let $g(x) = x^2$ with domain $[0, b]$. Let X be the region bounded by $0 \leq x \leq b$, and between the x-axis and the graph of g.





We pick a positive integer n (which we will later take to ∞), and divide the interval $[0, b]$ into n non-overlapping subintervals of length $\frac{b}{n}$. The partition points are

$$0,\ \ \frac{b}{n},\ \ 2\,\frac{b}{n},\ \ 3\,\frac{b}{n},\ \ \ldots\ \ ,\ \ k\,\frac{b}{n},\ \ \ldots\ ,\ \ (n-1)\,\frac{b}{n},\ \ n\,\frac{b}{n} \;=\; b$$

For the subregion X_k, which lies above the subinterval $\mathcal{I}_k = [(k-1)(\dfrac{b}{n}), k(\dfrac{b}{n})]$, we have

$$\left\{ \begin{array}{c} \text{area of the rectangle of} \\ \text{height } g\big((k-1)\,(\frac{b}{n})\big) \end{array} \right\} \leq \text{area}(X_k) \leq \left\{ \begin{array}{c} \text{area of the rectangle of} \\ \text{height } g\big(k\,(\frac{b}{n})\big) \end{array} \right\}$$

$$\frac{b}{n} \cdot (\,(k-1)\,(\frac{b}{n})\,)^2 \leq \text{area}(X_k) \leq \frac{b}{n} \cdot (\,k\,(\frac{b}{n})\,)^2$$

$$(k-1)^2\,\frac{b^3}{n^3} \leq \text{area}(X_k) \leq k^2\,\frac{b^3}{n^3}$$

We add together all the areas of the subregions X_k to get:

$$\Big(\sum_{k=1}^{n} (k-1)^2\Big)\,\frac{b^3}{n^3} \leq \text{area}(X) \leq \Big(\sum_{k=1}^{n} k^2\Big)\,\frac{b^3}{n^3}$$

The area of X is trapped between the two sides. We will let $n \to \infty$, and see that area(X) is trapped below by $\frac{b^3}{3}$ and above by $\frac{b^3}{3}$ and therefore area$(X) = \frac{b^3}{3}$.
In the above, we see there is a summation of squares. There is a formula for the summation of the squares $1^2 + 2^2 + 3^2 + \cdots + r^2$. It is:

$$\sum_{j=1}^{r} j^2 = \frac{r\,(r+1)\,(2r+1)}{6}\,.$$

So, we get

$$\sum_{k=1}^{n} (k-1)^2 = \sum_{j=1}^{n-1} j^2 = \frac{(n-1)\;((n-1)+1)\;(2(n-1)+1)}{6} = \frac{(n-1)\;(n)\;(2n-1)}{6}$$

and

$$\sum_{k=1}^{n} k^2 = \frac{(n)\;(n+1)\;(2n+1)}{6}\,.$$

So,

$$\frac{(n-1)\;(n)\;(2n-1)}{6}\,\frac{b^3}{n^3} \leq \text{area}(X) \leq \frac{(n)\;(n+1)\;(2n+1)}{6}\,\frac{b^3}{n^3}$$

$$\frac{(n-1)}{n}\;\frac{n}{n}\;\frac{2n-1}{2n}\;\frac{b^3}{3} \leq \text{area}(X) \leq \frac{n}{n}\;\frac{n+1}{n}\;\frac{2n+1}{2n}\;\frac{b^3}{3}$$

We now take the limit as $n \to \infty$ to get

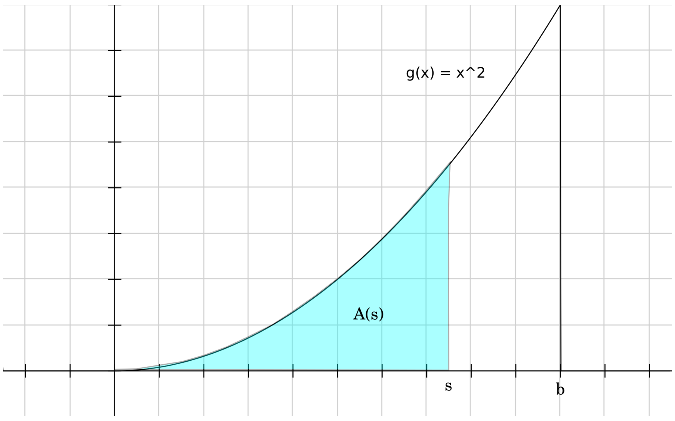
$$\frac{b^3}{3} \leq \text{area}(X) \leq \frac{b^3}{3}\,;$$

so, area$(X) = \frac{b^3}{3}$.

The area function associated to the function $g(x) = x^2$.

A very important insight found by the discoverers of calculus (Sir Issac Newton and Gottfried Leibniz) is that it is not just important to find the area of a shape, but to consider the area function associated to a function. For our example function $g(x) = x^2$, with domain $[0, b]$ we define the area function

$$A(s) = \text{area of the region between the } x\text{-axis and the graph of } g \text{ over the interval } [0, s]$$



The answer is gotten by replacing the interval $[0, b]$ with $[0, s]$. We get

$$A(s) = \frac{s^3}{3}, \quad \text{with domain } [0, b].$$

Notice $A(0) = 0$, $A(b) = \frac{b^3}{3}$, and notice very importantly that $A'(x) = x^2$. We see in this particular example, the area function A is an anti-derivative for the function g.