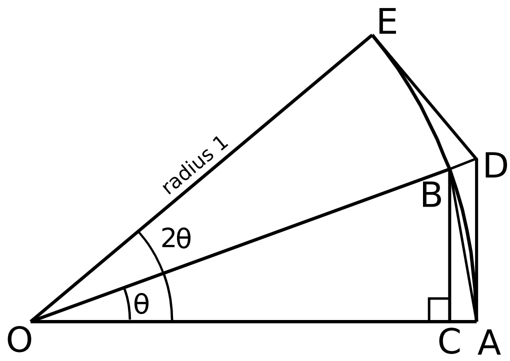
Consider a circular sector with center O and radius 1. Let A, B , E be points on the circular sector as shown.



Then:

$$\text{length}(BC) = \sin(\theta), \qquad \text{length}(\text{Arc(AB)}) = \theta, \qquad \text{and} \qquad \text{length}(AD) = \tan(\theta)\ .$$

14.1 1st inequalties and the limit $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

For $0 < \theta < \frac{\pi}{4}$, so 2θ is at most $\frac{\pi}{2}$ (90 degrees), we have:

$$\sin(\theta) = \text{length}(BC) < \text{length}(AB) < \text{length}(\text{Arc(AB)}) = \theta\ ,$$

and

$$2\theta = 2\,\text{length}(\text{Arc(AB)}) = \text{length}(\text{ Arc(AE)}) < \text{length}(AD) + \text{length}(DE) = 2\tan(\theta)\ .$$

In summary, we have:

$$\sin(\theta) < \theta < \tan(\theta)$$

We manipulate to:

$$\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1\ , \text{ valid for } \ 0 < \theta < \frac{\pi}{4}.$$

The three functions $\cos(\theta)$, $\frac{\sin(\theta)}{\theta}$, and 1 are all even, so the inequality is also true for $-\frac{\pi}{4} < \theta < 0$. In a picture:

The rule $\dfrac{\sin(\theta)}{\theta}$ does NOT allow input of $\theta = 0$. But, the function $\dfrac{\sin(\theta)}{\theta}$ is caught (for $\theta \neq 0$) between the two functions $\cos(\theta)$ and 1. We can apply the squeeze theorem to get

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \; = \; 1 \, .$$

14.2 2nd limit involving the function $\frac{1-\cos(\theta)}{\theta}$

From the 1st picture we have:

$$\text{length}(AC)^2 \; + \; \text{length}(BC)^2 \; = \; \text{length}(AB)^2 \; \leq \; \text{length}(\text{Arc(AB)})^2 \; , \quad \text{so}$$

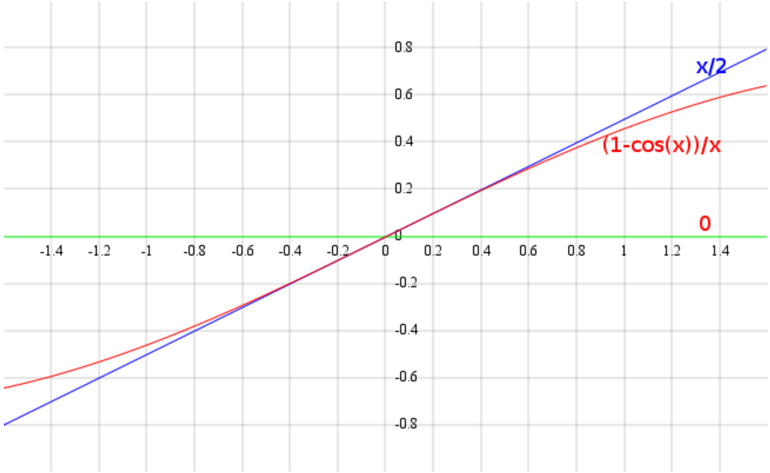
$$(1 - \cos(\theta))^2 \; + \; (\sin(\theta))^2 \; \leq \; \theta^2 \; , \qquad \text{then expand to get}$$

$$2 \, (\; 1 \; - \; \cos(\theta) \;) \; \leq \; \theta^2 \; , \qquad \text{and deduce}$$

$$0 \; \leq \; \frac{(\; 1 \; - \; \cos(\theta) \;)}{\theta} \; \leq \; \frac{\theta}{2} \, , \qquad \text{for } 0 \; < \theta \; < \frac{\pi}{4}$$

The (odd symmetry) rule $\dfrac{(\; 1 - \cos(\theta) \;)}{\theta}$ does not allow input $\theta = 0$.

A picture of this rule (graph in red), with values of $\frac{(\ 1\ -\ \cos(\theta)\ )}{\theta}$ caught between 0 (graph in yellow) and $\frac{\theta}{2}$ (graph in blue), is:



As before, we can deduce

$$\lim_{\theta \to 0} \frac{(\ 1\ -\ \cos(\theta)\ )}{\theta} \ = \ 0\,.$$

These two limits $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \ = \ 1$, and $\lim_{\theta \to 0} \frac{(\ 1\ -\ \cos(\theta)\ )}{\theta} \ = \ 0$ are very important. They are used later to compute the slope of tangent lines to the functions sin and cos.

15 Tangent slope of the functions sine and cosine

15.1 Tangent slope to the graph of sine at the point $(b, \sin(b))$.

The tangent slope at the graph point $(b, \sin(b))$ is the limit of the difference quotient:

$$\frac{\sin(\,b + h\,)\ -\ \sin(\,b\,)}{h}\,.$$

We use the formula for the sine of a sum to get:

$$\begin{aligned} \sin(\,b + h\,)\ -\ \sin(\,b\,) \ &= \ \sin(b)\ \cos(h)\ +\ \cos(b)\ \sin(h)\ -\ \sin(b) \\ &= \ \sin(b)\,(\,\cos(h)\ -\ 1\,)\ +\ \cos(b)\ \sin(h) \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0} \frac{\sin(\,b + h\,)\ -\ \sin(\,b\,)}{h} \ &= \ \lim_{h \to 0} \left(\sin(b)\, \frac{(\cos(h)\ -\ 1)}{h}\ +\ \cos(b)\, \frac{\sin(h)}{h} \right) \\ &= \ \sin(b)\ \lim_{h \to 0} \frac{(\cos(h)\ -\ 1)}{h}\ +\ \cos(b)\ \lim_{h \to 0} \frac{\sin(h)}{h} \\ &= \ \sin(b)\ 0\ +\ \cos(b)\ 1 \\ &= \ \cos(b) \end{aligned}$$

The tangent slope to the graph of sine at the point $(b, \sin(b))$ is $\cos(b)$.

15.2 Tangent slope to the graph of cosine at the point $(b, \cos(b))$.

The tangent slope at the graph point $(b, \cos(b))$ is the limit of the difference quotient:

$$\frac{\cos(\, b + h\,) \;-\; \cos(\, b\,)}{h}\;.$$

We use the formula for the cosine of a sum to get:

$$\begin{aligned}\cos(\, b + h\,) \;-\; \cos(\, b\,) &= \cos(b)\, \cos(h) \;-\; \sin(b)\, \sin(h) \;-\; \cos(b)\\ &= \cos(b)\,(\,\cos(h) \;-\; 1\,) \;-\; \sin(b)\, \sin(h)\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(\, b + h\,) \;-\; \cos(\, b\,)}{h} &= \lim_{h\to 0} \left(\cos(b)\, \frac{(\,\cos(h) \;-\; 1\,)}{h} \;-\; \sin(b)\, \frac{\sin(h)}{h} \right)\\ &= \cos(b)\, \lim_{h\to 0} \frac{(\,\cos(h) \;-\; 1\,)}{h} \;-\; \sin(b)\, \lim_{h\to 0} \frac{\sin(h)}{h}\end{aligned}$$

Our earlier calculation using the squeeze theorem showed

$$\lim_{h\to 0} \frac{(\,\cos(h) \;-\; 1\,)}{h} \;=\; 0 \quad \text{and} \quad \lim_{h\to 0} \frac{\sin(h)}{h} \;=\; 1\;;$$

so,

$$\lim_{h\to 0} \frac{\cos(\, b + h\,) \;-\; \cos(\, b\,)}{h} \;=\; \cos(b)\, 0 \;-\; \sin(b)\, 1 \;=\; -\sin(b)$$

The tangent slope to the graph of cosine at the point $(b, \cos(b))$ is $-\sin(b)$.