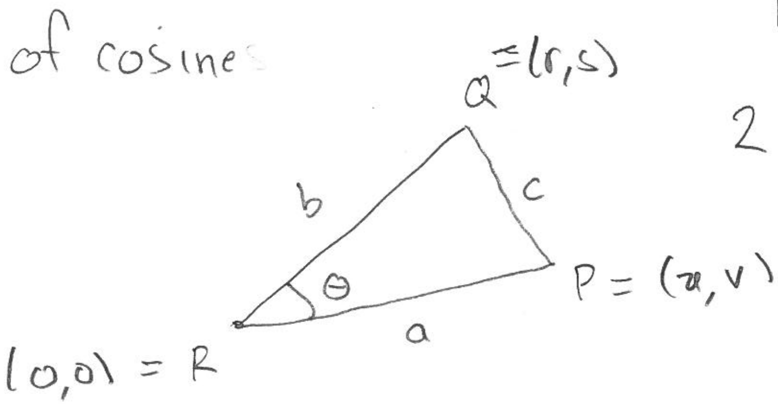
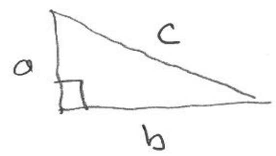
Week 3 W LO3 2pm

1

Law of cosines



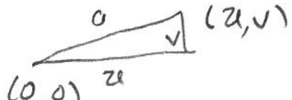
$2ab\cos\theta = a^2+b^2-c^2$



For right triangle
$c^2 = a^2+b^2$

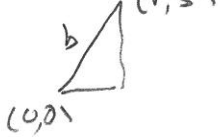
If we use coordinates and assume $R=(0,0)$

$P=(u,v)$ $\overline{RP}$



$a^2 = u^2+v^2$

$Q=(r,s)$ $\overline{RQ}$

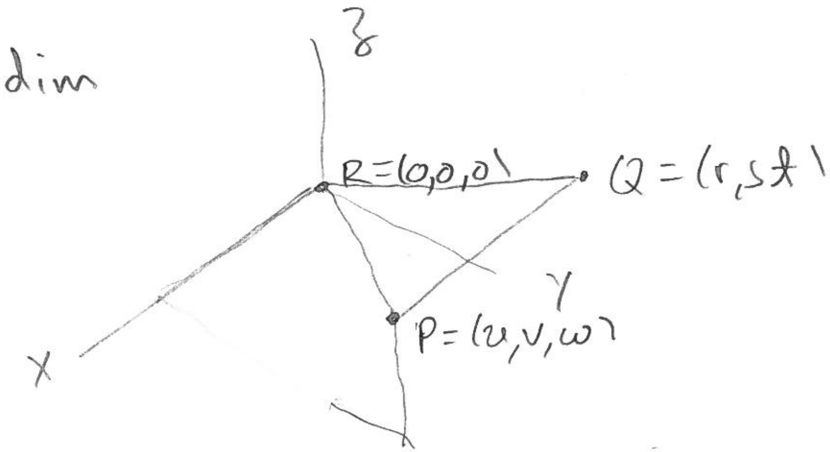


$b^2 = r^2+s^2$

$\overline{PQ}$ $c^2 = (r-u)^2+(s-v)^2$

$\therefore$ $2ab\cos\theta = \frac{1}{2}(a^2+b^2-c^2) = \cdots = (ru+sv)$ quantity called dot product

In 3-dim



dot product of (u,v,w) and (r,s,t)

$ur+vs+wt$

$= ab\cos\theta$. (Law of cosines)

Example $\quad R = (0,0,0)$

$P = (2,2,1) \qquad |RP| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$

$Q = (1,2,2) \qquad |RQ| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3.$

Dot product of P and Q is $(2\cdot1) + (2\cdot2) + (1\cdot2) = 8.$

Law of cosines says $\quad 3\cdot 3\cos(\theta) = 8$

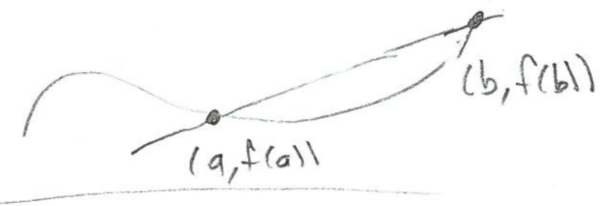
$$\cos(\theta) = \frac{8}{3\cdot3} = \frac{8}{9}. \qquad \theta = \arccos\left(\frac{8}{9}\right) = 0.4759 \text{ radians}$$

$$= 0.4759 \text{ radians}\left(\frac{360^\circ}{2\pi \text{ radians}}\right)$$

$$= 27.26^\circ.$$

Secant Line

If f is function and $(a, f(a))$, $(b, f(b))$ $a \neq b$ are distinct points on graph



The line through $(a, f(a))$ and $(b, f(b))$ is called secant line.

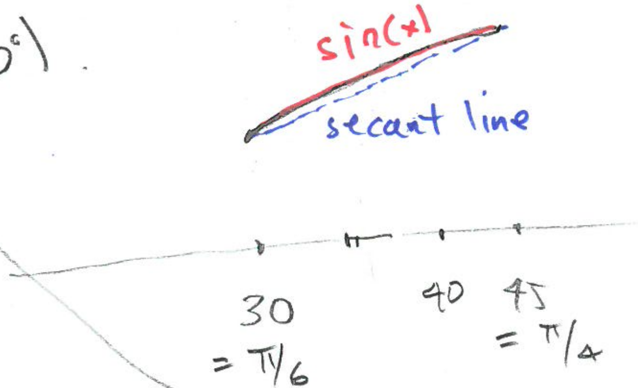
Uses of secant line

① Interpolation. Use secant line to find approximate values to function for input between a and b.

Example Sine function $\sin(30°) = \frac{1}{2} = 0.500$

$\sin(45°) = \frac{\sqrt{2}}{2} = 0.707\ldots$

Find estimate for $\sin(40°)$.



40° is $\frac{2}{3}$ way from 30° to 45°.

Value of secant line at 40° $\frac{2}{3}$ way from 0.500 to 0.707.

$$\sin(40°) \doteq \sin(30°) + \frac{2}{3}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$

$$= \frac{1}{2} + \frac{2}{3}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)$$

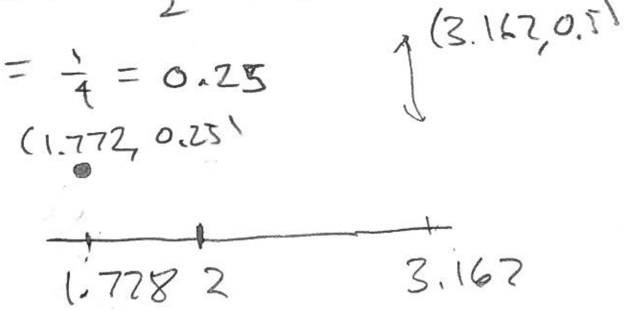
$$= 0.638$$ estimate

true value $\sin(40°) = 0.643$.

Example Estimate $\log_{10}(2)$.

Things we know $\sqrt{10} = 3.162$ so $\log_{10}(3.162) = \frac{1}{2} = 0.5$

$\sqrt{\sqrt{10}} = \sqrt{3.162} = 1.778$ so $\log_{10}(1.778) = \frac{1}{4} = 0.25$



Number 2 is between 1.778 and 3.162.

2 is $\dfrac{2 - 1.778}{3.162 - 1.778}$ of the way from 1.778 to 3.162.

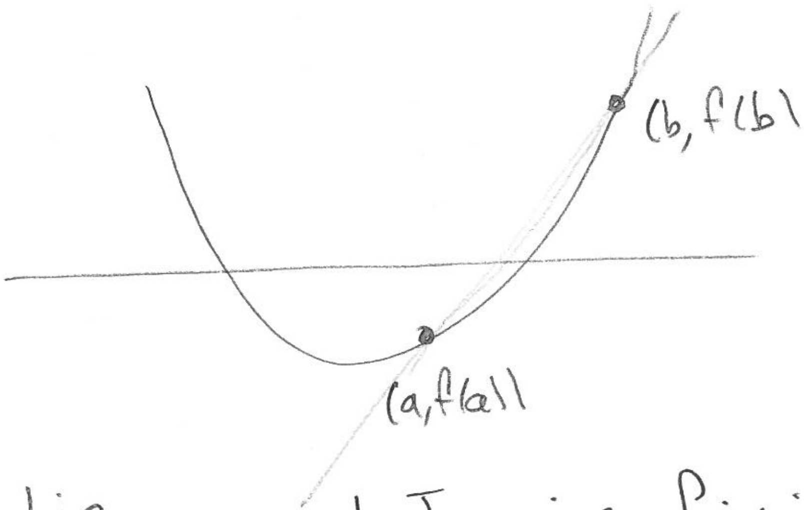
$\log_{10}(2)$ is same % of way from 0.25 to 0.50

We estimate $\log_{10}(2) \doteq 0.25 + \left(\dfrac{2 - 1.778}{3.162 - 1.778}\right)(0.5 - 0.25)$

$\doteq 0.290$.

Actual value of $\log_{10}(2)$ is 0.301

Secant slope



slope of secant line

$$m = \frac{f(b)-f(a)}{b-a}$$

Imagine fixing input a, allow input b to vary. Get function of b

$$m(b) = \frac{f(b)-f(a)}{b-a}$$ (domain NOT allowed to take b equal to a)

Then let variable b approach the fixed input a. In good situations secant line (secant slope) approaches tangent line (tangent slope).

Do same thing for general fixed input a.

$m(b) =$ secant slope $= \dfrac{f(b) - f(a)}{b-a}$ (think a fixed, b variable)

$= \dfrac{(\frac{1}{4}b^2 - b) - (\frac{1}{4}a^2 - a)}{b-a}$ (domain is $\{b \in \mathbb{R} \mid b \neq a\}$).

$= \dfrac{\frac{1}{4}(b^2 - a^2) - (b-a)}{(b-a)} = \dfrac{\frac{1}{4}(b+a)\cancel{(b-a)} - \cancel{(b-a)}}{\cancel{(b-a)}}$

$m(b) = \frac{1}{4}(b+a) - 1$ This is secant slope.

From our formula for secant slope we see that as $b \to a$, that

$$\lim_{b \to a} m(b) = \lim_{b \to a} \frac{1}{4}(b+a) - 1 = \frac{1}{4}(a+a) - 1 = \frac{a}{2} - 1.$$

This is tangent slope at graph point $(a, f(a))$.

For $a = 4$, we get tangent slope at $(4, f(4))$ is $\frac{4}{2} - 1 = 2 - 1 = 1$

$a = 0$, ———— $(0, f(0))$ is $\frac{0}{2} - 1 = 0 - 1 = -1$

$a = 2$, ———— $(2, f(2))$ is $\frac{2}{2} - 1 = 1 - 1 = 0$

Example $f(x) = \frac{1}{4}x^2 - x$

Take $a = 3$ (fixed). Secant slope $(3, f(3))$ to $(b, f(b))$ is

$$m(b) = \frac{\left(\frac{1}{4}b^2 - b\right) - \left(\frac{9}{4} - 3\right)}{(b-3)} \qquad \text{domain is } \{b \neq 3\}.$$

$$= \frac{\frac{1}{4}(b^2 - 9) - (b-3)}{(b-3)} = \frac{\frac{1}{4}(b+3)(b-3) - (b-3)}{(b-3)}$$

$$m(b) = \frac{1}{4}(b+3) - 1$$

We let $b \longrightarrow 3$

$$m(b) = \frac{1}{4}(b+3) - 1 \rightarrow \frac{1}{4}(3+3) - 1$$

$$= \frac{6}{4} - 1 = \frac{1}{2}$$

We have $\displaystyle\lim_{b \to 3} m(b) = \lim_{b \to 3} \frac{1}{4}(b+3) - 1 = \frac{6}{4} - 1 = \frac{1}{2}$

The tangent slope at $(3, f(3))$ is $\frac{1}{2}$.