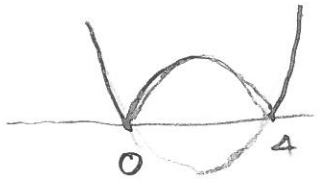
Tangent slope as limit of secant slopes

Example $f(x) = \left|\frac{1}{4}x^2 - x\right|$

$\frac{1}{4}x^2 - x = x\left(\frac{1}{4}x - 1\right)$

roots 0, 4.



$$= \begin{cases} \frac{1}{4}x^2 - x & \text{when } x<0 \text{ OR } 4<x \\ -\left(\frac{1}{4}x^2 - x\right) & \text{when } 0 \le x \le 4. \end{cases}$$

Suppose $a \neq 0$ or 4, and suppose b is <u>near</u> a (same interval).

$$m(b) = \frac{f(b)-f(a)}{b-a} = \frac{\left(\frac{1}{4}b^2 - b\right) - \left(\frac{1}{4}a^2 - a\right)}{b-a}$$

domain of b is $b \neq a$

$$= \frac{\frac{1}{4}(b^2-a^2) - (b-a)}{b-a} = \text{algebra} = \frac{1}{4}(b+a) - 1$$ secant slope

As $b \longrightarrow a$ we get $\lim_{b\to a} m(b) = \lim_{b\to a}\left(\frac{1}{4}(b+a) - 1\right) = \frac{1}{4}(a+a) - 1$

$= \frac{a}{2} - 1.$

Tangent slope at graph point $(a, f(a))$ is $\frac{a}{2} - 1$

What happens at $a=0$? $f(0) = \frac{1}{4}0^2 - 0 = 0$

$$m(b) = \frac{f(b) - \cancel{f(0)}^{0}}{b - \cancel{0}} = \frac{f(b)}{b} \quad (\text{domain } b \neq 0).$$

$$= \begin{cases} \dfrac{-(\frac{1}{4}b^2 - b)}{b} & \text{when } b > 0 \\ \dfrac{(\frac{1}{4}b^2 - b)}{b} & \text{when } b < 0 \end{cases}$$

$$= \begin{cases} -\frac{1}{4}b + 1 & \text{when } b > 0 \\ \frac{1}{4}b - 1 & \text{when } b < 0 \end{cases}$$

as $b \to 0$ and $b > 0$, $\lim\limits_{b \to 0^+} m(b) = \lim\limits_{b \to 0^+} (-\frac{1}{4}b + 1) = 0 + 1 = 1$

as $b \to 0$ and $b < 0$ $\lim\limits_{b \to 0^-} m(b) = \lim\limits_{b \to 0^-} (\frac{1}{4}b - 1) = 0 - 1 = -1$

$\lim\limits_{b \to 0} m(b)$ does not exist

At $a=0$ there is NO tangent slope

General limits

Situation.

1. An interval I and an approach point $a \in I$
2. function f with domain $\{ x \in I \mid x \neq a \}$.

We say $\lim\limits_{x \to a} f(x) = L$ if by taking input values x near/close to a (but not equal), the output values $f(x)$ are near/close to the number L.

Examples. 1. $I = \{ 1 \leq x \leq 3 \}$, approach point $a = 2$

$$f(x) = \frac{x^2 - 2^2}{x - 2} = \text{secant slope from pt } (2, 2^2) \text{ to } (x, x^2)$$

domain $x \neq 2$.

$$\frac{x^2 - 2^2}{x-2} = \frac{(x+2)(x-2)}{(x-2)}$$

$$\lim_{x \to 2} \left(\frac{x^2 - 2^2}{x - 2} \right) = \lim_{x \to 2} (x + 2) = 4.$$

(2) WW3 #7. $\lim_{h \to 0} \dfrac{\frac{6}{a+h} - \frac{6}{a}}{h}$.

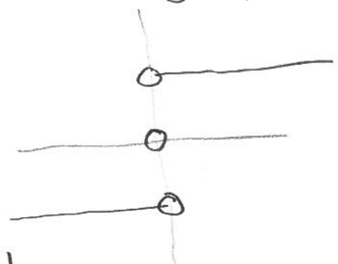
$f(h) = \dfrac{\frac{6}{a+h} - \frac{6}{a}}{h}$ with approach point 0.

domain of f is $\{h \in \mathbb{R} \mid h \neq 0, h \neq -a\}$.

$$= \frac{\frac{6a - 6(a+h)}{(a+h)a}}{h} = \frac{\cancel{6a} - \cancel{6a} - 6h}{h(a+h)a} = \frac{-6\cancel{h}}{\cancel{h}(a+h)a} = \frac{-6}{(a+h)a}.$$

$$\lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{-6}{(a+h)a} = \frac{-6}{a \cdot a} = \frac{-6}{a^2}.$$

graph of $\frac{|x|}{x}$



(3) $I = \{-1 \leq x \leq 1\}$ approach 0.

$f(x) = \dfrac{|x|}{x}$ (domain $\{x \in I \mid x \neq 0\}$.

To see/determine $\lim_{x \to 0} \frac{|x|}{x}$ <u>does not exists</u>

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

④ $I = \{-1 \le x \le 1\}$ approach point 0

$f(x) = \sin\left(\frac{1}{x}\right)$ domain $= \{x \in I \mid x \neq 0\}$.

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exists

⑤ $\lim_{x \to 0} \frac{1}{x}$ does not exist $\lim_{x \to 0^+} \left(\frac{1}{x}\right) = +\infty$

$\lim_{x \to 0^-} \left(\frac{1}{x}\right) = -\infty$.

⑥ $\lim_{x \to 0} \frac{\sin(x)}{x} = \underline{1}$ (true graphically). Proof next week.

This is important limit. It is used to find tangent slope of sine and cosine functions.

One sided limits. Restrict our input variable to approach the approach point from only above/right side or below/left side. We use notation $x \to a^+$

$x \to a^-$

Relationship to two side limit

$$\lim_{x \to a} f(x) = L \iff \text{BOTH} \lim_{x \to a^+} f(x) = L$$

$$\text{AND} \lim_{x \to a^-} f(x) = L.$$

Example $\lim_{x \to 0^+} \frac{|x|}{x} = 1$, $\lim_{x \to 0^-} \frac{|x|}{x} = -1$, so $\lim_{x \to 0} \frac{|x|}{x} = \text{DNE}$