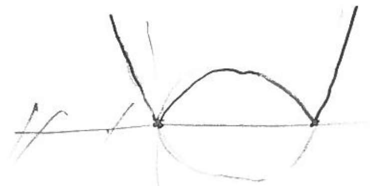
Tangent slope as limit of secant slopes.

<u>Example</u> $f(x) = \left|\frac{1}{4}x^2 - x\right|$

$\frac{1}{4}x^2 - x = x\left(\frac{1}{4}x - 1\right)$

roots at $x=0$, $x=4$



$$= \begin{cases} \frac{1}{4}x^2 - x & \text{when } x<0 \text{ or } 4<x \\ -\left(\frac{1}{4}x^2 - x\right) & \text{when } 0 \le x \le 4. \end{cases}$$

Secant slope $a \neq 0$ or 4, and b <u>close</u> to a (same interval as a)

When $a<0$ (so assume $b<0$) we

$$m(b) = \frac{\left(\frac{1}{4}(b^2) - b\right) - \left(\frac{1}{4}a^2 - a\right)}{b-a} \quad (\text{domain is } b \neq a)$$

$$= \frac{\frac{1}{4}(b^2 - a^2) - (b-a)}{(b-a)} = \frac{1}{4}(b+a) - 1.$$

$$\lim_{b\to a} m(b) = \lim_{b\to a} \frac{1}{4}(b+a) - 1 = \frac{1}{4}(a+a) - 1 = \frac{a}{2} - 1$$ is tangent slope at $(a, f(a))$.

For $0 < a < 4$ (a fixed) and b near a, we have

$$m(b) = \frac{-\left(\frac{1}{4}b^2 - b\right) - \left(-\left(\frac{1}{4}a^2 - a\right)\right)}{b-a}$$

$$= -\left(\frac{1}{4}(b+a) - 1\right) \quad \text{secant slope}$$

$$\lim_{b \to a} m(b) = \lim_{b \to a} -\left(\frac{1}{4}(b+a) - 1\right) = -\left(\frac{1}{4}(a+a) - 1\right) = -\left(\frac{1}{2}a - 1\right).$$

minus comes from taking absolute value.

What about 0? Here $f(0) = 0$.

$$m(b) = \begin{cases} \dfrac{-\left(\frac{1}{4}b^2 - b\right) - 0}{b} & \text{when } b > 0 \\[2ex] \dfrac{\left(\frac{1}{4}b^2 - b\right) - 0}{b} & \text{when } b < 0 \end{cases}$$

$$= \begin{cases} -\frac{1}{4}b + 1 & \text{when } b > 0 \\ \frac{1}{4}b - 1 & \text{when } b < 0. \end{cases}$$

$$\lim_{b \to 0^+} m(b) = \lim_{b \to 0^+} \left(-\frac{1}{4}b + 1\right) = -\frac{1}{4}0 + 1 = 1$$

$$\lim_{b \to 0^-} m(b) = \lim_{b \to 0} \left(\frac{1}{4}b - 1\right) = \frac{1}{4} \cdot 0 - 1 = -1$$

NO TANGENT SLOPE.

General limits

Situation

(1) some interval I, and an approach point $a \in I$.

(2) function f with domain $\{x \in I \mid x \neq a\}$.

We say $\lim_{x \to a} f(x) = L$ if by taking input x close a the output $f(x)$ is close the number L.

Examples (1) $I = \{1 \leq x \leq 3\}$, $f(x) = \dfrac{x^2 - 2^2}{x-2} = \dfrac{(x+2)(x-2)}{(x-2)} = x+2$

approach point $a = 2$.

$$\lim_{x \to 2} \frac{x^2-2^2}{x-2} = \lim_{x \to 2} (x+2) = 2+2 = 4. \qquad \lim_{x \to 2} \frac{x^2-2^2}{x-2} = 4$$

(2) WW3 #7. $f(h) = \dfrac{\frac{6}{a+h} - \frac{6}{a}}{h}$, interval is $\{h \mid h \neq -a \text{ and } h \neq 0\}$.

secant slope of $\frac{6}{x}$ for inputs $a, b = a+h$.

$$\frac{\frac{6}{a+h} - \frac{6}{a}}{h} = \frac{\frac{6a - 6(a+h)}{(a+h)a}}{h} = \frac{\frac{-6\cancel{h}}{(a+h)a}}{\cancel{h}} = \frac{-6}{(a+h)a}.$$

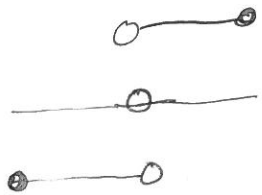
Find $\lim_{h \to 0} f(h)$.

$$= \lim_{h \to 0} \frac{-6}{(a+h)a} = \frac{-6}{(a) \cdot a} = \frac{-6}{a^2}$$

Tangent slope of $\frac{6}{x}$ at input $x = a$ is $-6/a^2$.

More examples ③ $I = \{-1 \le x \le 1\}$ approach point $a=0$

$f(x) = \frac{|x|}{x}$ has domain $\{x \in I \mid x \neq a = 0\}$.



$\lim_{x \to 0} \frac{|x|}{x}$ does not exists.

Because there is no single L so that

$\frac{|x|}{x}$ is close L when x small (close to 0).

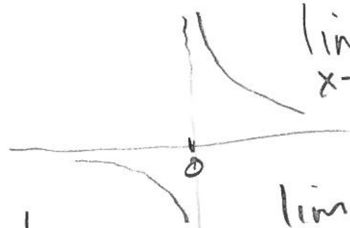
④ $I = \{-1 \le x \le 1\}$. approach point $a=0$

$f(x) = \sin\left(\frac{1}{x}\right)$ has domain $\{x \in I \mid x \neq a\}$.

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exists

⑤ $I = \{-1 \le x \le 1\}$. approach point $a=0$

$f(x) = \frac{1}{x}$ domain $\{x \in I \mid x \neq a\}$.



$\lim_{x \to 0} \frac{1}{x}$ DNE

$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

(6) $I = \{-1 \le x \le 1\}$ $a = 0$ approach point

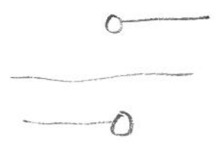
$f(x) = \frac{\sin(x)}{x}$ $\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$ Very important limit. Use to find tangent slopes of sine and cosine functions.

1

One-sided limits. Interval I, approach point a.

Only difference is restrict ourself to approach from only one side. (right/above) or (left/below).

Example $f(x) = \frac{|x|}{x}$ (domain $x \neq 0$). $\lim_{x \to 0^+} \frac{|x|}{x} = 1$



$\lim_{x \to 0^-} \frac{|x|}{x} = -1$

Relationship to limit.

$\lim_{x \to a} f(x) = L \iff$ BOTH $\lim_{x \to a^+} f(x) = L$

$\lim_{x \to a^-} f(x) = L.$

WW3 # 3

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x-5) = \text{"}-1^-\text{"} - 5 = -6.$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2+3) = (-1)^2 + 3 = 4$$

$\lim_{x \to -1} f(x) = \text{DNE}$ since one-sided limits are $-6, 4$ which are not equal.

$\lim_{x \to 1} f(x)$ need to check $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (5-x) = 5-1 = 4$

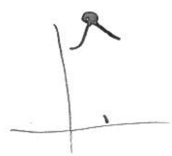
$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2+3) = 1^2+3 = 4$

yes limit is 4

$f(-1) = -1-5 = -6.$

$f(1) = 1^2+3 = 4.$

NOTE $\lim_{x \to 1} f(x) = 4$ AND $f(1) = 4$



f is continuous at

$x=4$