<u>Limits</u> Intuition

$\lim_{x \to a} f(x) = L$ means the output $f(x)$ of the function is near L when input x is near (but not equal) to a.

Do <u>NOT</u> care whether a is allowed as input to f

Example WW3 #8 Find $\lim_{t \to 5^+} \frac{|25-t^2|}{5-t}$

$t \to 5^+$ the variable t approaches 5, with extra condition $t > 5$.
When $t > 5$, then $(25-t^2) < 0$ so $|25-t^2| = t^2 - 25$. So

$$\frac{|25-t^2|}{5-t} = \frac{t^2-25}{5-t} = \frac{(t-5)(t+5)}{(5-t)} = -(t+5) \quad (\text{assumption } t > 5).$$

So $\lim_{t \to 5^+} \frac{|25-t^2|}{5-t} = \lim_{t \to 5^+} -(t+5) = -(5+5) = -10.$

WW3 #9 Find $\lim\limits_{x \to \frac{4}{5.5}} \left(\frac{5.5x - 4}{\sqrt{5.5x} - 2} \right)$ Function does NOT allow input $x = \frac{4}{5.5}$

Assume $x \neq \frac{4}{5.5}$. Manipulate expression for function.

Goal is to manipulate to something from which we can see limit

$$\left(\frac{5.5x - 4}{\sqrt{5.5x} - 2} \right) \cdot \left(\frac{\sqrt{5.5x} + 2}{\sqrt{5.5x} + 2} \right) = \frac{\cancel{(5.5x - 4)} (\sqrt{5.5x} + 2)}{\cancel{(5.5x - 4)}} = (\sqrt{5.5}\, x + 2)$$

So $\lim\limits_{x \to \frac{4}{5.5}} \left(\frac{5.5x - 4}{\sqrt{5.5x} - 2} \right) = \lim\limits_{x \to \frac{4}{5.5}} (\sqrt{5.5x} + 2) = (\sqrt{4} + 2) = (2 + 2).$

Quantitative definition of limit

$\lim_{x \to a} f(x) = L$. So say output is near L, is to be given "tolerance" T, and we are challenged to make the outputs within T of L.

In terms of inequality $|f(x) - L| < T$.

Example "I challenge you to make outputs within 10^{-6} of L".

$|f(x) - L| < 10^{-6}$.

You would need to find how close x should be to a so you meet the challenge. Find a "reply" R so that

$|x - a| < R \Longrightarrow |f(x) - L| < T$.

within R of a

Example. $\lim_{x \to 3} 4x = 12$. If I challenge you to make output $f(x) = 4x$ within 10^{-6}, you need to find reply R so that $|x-3| < R$ automatically insures $|4x - 12| < 10^{-6}$. want $|x-3| < \frac{10^{-6}}{4}$. Take this as R.

If $|x-3| < \frac{10^{-6}}{4}$, then $|4x-12| < 10^{-6}$

output $f(x)=4x$ is within 10^{-6} of 12.

If the challenge is to make $|4x-12| < 10^{-9}$, we can do so by $|x-3| < \frac{10^{-9}}{4}$. If the challenge is to make $|4x-12| < 10^{-100}$ we $|x-3| < \frac{10^{-100}}{4}$.

WW3 #1. $f(x) = \frac{x^2-4}{x-2}$. $\lim_{x\to 2} f(x) = 2+2 = 4$

$= (x+2) \quad x \neq 2.$

Challenge.

$|f(x)-4| < .01$. $|(f(x)-4)| = |(x+2)-4| = |x-2|$

How close to 2 should we be? We need $|x-2| < .01$

If $|x-2| < 0.01$, then $|f(x)-4| < 0.01$

Show $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ using quantitative definition.

Challenge you to make $|\sqrt{x} - \sqrt{a}| < T$.

How close should x be to a?

$$|\sqrt{x} - \sqrt{a}| = \left| (\sqrt{x} - \sqrt{a}) \frac{(\sqrt{x}+\sqrt{a})}{(\sqrt{x}+\sqrt{a})} \right| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}. \quad \text{want} < T.$$

Note

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{a}}.$$

If we make $\frac{|x-a|}{\sqrt{a}} < T$, we get $|\sqrt{x} - \sqrt{a}| < T$ which was the challenge.

So take $|x-a| < T\sqrt{a}$. $R = T\sqrt{a}$.

If $T = 10^{-6}$, $a = 100$, then $|\sqrt{x} - 10| < 10^{-6}$ when $|x-100| < 10^{-6} \cdot 10$
$\sqrt{a} = 10$ $< 10^{-5}$

Rules for computing limits.

Suppose $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, then

<u>sum rule</u> $\lim_{x\to a} (f(x)+g(x)) = L+M$

<u>product rule</u> $\lim_{x\to a} (f(x)g(x)) = L\cdot M$

<u>quotient rule</u> if $M\neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

<u>Composition rule</u> If $\lim_{x\to a} g(x) = b$, $\lim_{y\to b} f(y) = L$

then composite $f(g(x))$. satisfies

$$\lim_{x\to a} f(g(x)) = L.$$

WW3 #4. Assume $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = -8$, $\lim_{x\to a} h(x) = 10$.

(1). $\lim_{x\to a} (f(x)+g(x)) = 0+(-8) = -8$

(3). $\lim_{x\to a} (f(x)g(x)) = 0\cdot(-8) = 0$.

(5) $\lim_{x\to a} \left(\frac{f(x)}{h(x)}\right) = \frac{0}{10} = 0$.

(6) $\lim_{x\to a} \left(\frac{h(x)}{f(x)}\right) = \cancel{\frac{10}{0}}$ DNE

(7) $\lim_{x\to a} \sqrt{g(x)}$ = DNE because $\sqrt{-8}$ not allowed.

$\lim_{x\to a} \sqrt{h(x)} = \sqrt{10}$.

(8) $\lim_{x\to a} \frac{1}{g(x)-h(x)} = \frac{1}{(-8)-(10)} = -\frac{1}{18}$.

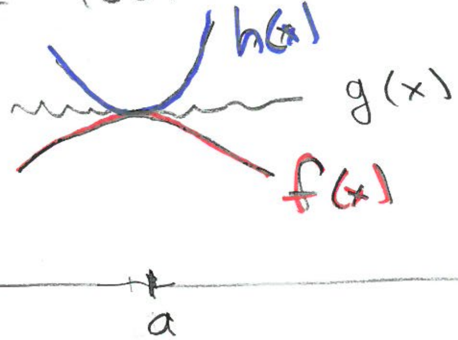
Squeeze Theorem for Limits   3 functions

$f(x) \le g(x) \le h(x)$ g "trapped" between f and h.

Squeeze Theorem

If $\lim_{x\to a} f(x) = L$, and $\lim_{x\to} h(x) = L$ (same limit)

then $\lim_{x\to a} g(x) = L$ too.



Example WW3 # 12   Given $10x - 26 \le f(x) \le x^2 + 6x - 22$

f trapped. What happens as $x \to 2$?

$\lim_{x\to 2} 10x - 26 = 10\cdot 2 - 26 = -6$, $\lim_{x\to 2} (x^2+6x-22) = 2^2 + 6\cdot 2 - 22$
$= 4 + 12 - 22$
$= -6$

By squeeze theorem $\lim_{x\to 2} f(x) = -6$.