Limits Intuition

$\lim_{x \to a} f(x) = L$ means for inputs x closed to the approach point (but not equal), the output $f(x)$ is close to L.

Ex WW3 # 8 Find $\lim_{t \to 5^+} \frac{|25 - t^2|}{5 - t}$ Note $\frac{|25-t^2|}{5-t}$ does NOT allow input $t = 5$.

The notation $t \to 5^+$ means t approaches 5, but is always bigger than 5. In this situation $(5-t) < 0$

$(25 - t^2) < 0 \therefore |25 - t^2| = t^2 - 25$

For $t > 5$ we have

$$\frac{|25 - t^2|}{5-t} = \frac{t^2 - 25}{5-t} = \frac{(t-5)(t+5)}{5-t} = -(t+5).$$

So $\lim_{t \to 5^+} \frac{|25-t^2|}{5-t} = \lim_{t \to 5^+} -(t+5) = -(5+5) = -10.$

WW3 #9 Find $\lim_{x \to \frac{4}{5.5}} \left(\frac{5.5x - 4}{\sqrt{5.5x} - 2} \right)$

Note ① the function $\left(\frac{5.5x - 4}{\sqrt{5.5x} - 2} \right)$ does NOT allow input $x = \frac{4}{5.5}$

② $x \to \frac{4}{5.5}$ means we can approach $\frac{4}{5.5}$ both above and below

$x \neq \frac{4}{5.5}$ we have

$$\left(\frac{5.5x - 4}{\sqrt{5.5x} - 2} \right) \cdot \left(\frac{\sqrt{5.5x} + 2}{\sqrt{5.5x} + 2} \right) = \frac{\cancel{(5.5x-4)}(\sqrt{5.5x}+2)}{\cancel{5.5x - 4}}$$

$$= \left(\sqrt{5.5x} + 2 \right) \text{ simpler expression}$$

Then $\lim_{x \to \frac{4}{5.5}} \left(\frac{5.5x - 4}{\sqrt{5.5x} - 2} \right) = \lim_{x \to \frac{4}{5.5}} \left(\sqrt{5.5x} + 2 \right) = \left(\sqrt{5.5 \tfrac{4}{5.5}} + 2 \right) = \left(\sqrt{4} + 2 \right)$

$= 2 + 2 = 4.$

Quantitative definition of limit: $\lim_{x \to a} f(x) = L$ means

if we present with a "tolerance" $T > 0$ and we challenge you to make $|f(x) - L| < T$ (make outputs of f within T of L),

then you can find a reply $R > 0$ so that

$$\underset{\text{within } R \text{ of } a}{0 < |x - a| < R} \xrightarrow[\text{gives}]{} |f(x) - L| < T.$$

Example ① $\lim_{x \to 5} 3x = 15$.

Using quantitative definition. Suppose I challenge you to make output values of $f(x)=3x$ within T of 15 provided $0<|x-5|<R$.

We want $0<|x-5|<R \implies |3x-15|<T$.

work backwards to find R. $|3x-15| = |3(x-5)| < T$ when

$$|x-5| < \frac{T}{3}$$

so we take $R = \frac{T}{3}$.

If $0<|x-5|<\frac{T}{3}$, then $|3(x-5)|<T$ so $|3x-15|<T$

② WW3 #1 $f(x)=\dfrac{x^2-4}{x-2}$ Claim $\lim_{x\to 2} f(x) = 4$.

Challenged to make $|f(x)-4|<0.01$

$\underbrace{\left|\left(\frac{x^2-4}{x-2}\right)-4\right|<.01}$

For $x \neq 2$, we simplify $\dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{(x-2)} = (x+2)$

re write as $|(x+2)-4|<.01$ which says $|x-2|<0.01$.

If $|x-2|<0.01$, then $\left|\left(\frac{x^2-4}{x-2}\right)-4\right|<.01$.

Example ③ Show $\lim_{x \to a} \sqrt{x} = \sqrt{a} \quad (a>0)$.

If I challenge you to make output $\sqrt{x}$ within T of $\sqrt{a}$, what should your reply be?

Want $|\sqrt{x} - \sqrt{a}| < T$ when $|x-a| < R$ (what is R?)

We use

$$|\sqrt{x} - \sqrt{a}| = \left|(\sqrt{x} - \sqrt{a}) \cdot \frac{(\sqrt{x}+\sqrt{a})}{(\sqrt{x}+\sqrt{a})}\right| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| \text{ want } < T.$$

But $\left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| < \left|\frac{x-a}{\sqrt{a}}\right|$. So if we make $\left|\frac{x-a}{\sqrt{a}}\right| < T$, then clearly

dropped $\sqrt{x}$

$$|\sqrt{x} - \sqrt{a}| = \left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right| < \left|\frac{x-a}{\sqrt{a}}\right| < T$$

So we can accomplish by $|x-a| < T\sqrt{a}$ This is R

So $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.

Important facts about limits If f, g two functions,

and $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

(1) <u>Sum rule</u> $\lim_{x \to a} (f(x) + g(x)) = (L + M)$

(2) <u>Product rule</u> $\lim_{x \to a} (f(x) g(x)) = L \cdot M$

(3) <u>Quotient rule</u> Provided $M \neq 0$, then $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}$

<u>Composition rule</u>. If f, g can be composed ($f \circ g$ make sense)

and $\lim_{x \to a} g(x) = b$, and $\lim_{y \to b} f(y) = K$, then

$$\lim_{x \to a} f(g(x)) = K$$

Examples WW3 #4

$\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x) = -8$, $\lim_{x\to a} h(x) = 10$.

(1) $\lim_{x\to a} (f(x)+g(x)) = 0+(-8) = -8$

(3) $\lim_{x\to a} (f(x)g(x)) = 0\cdot(-8) = 0$

(5) $\lim_{x\to a} \left(\frac{f(x)}{h(x)}\right) = \frac{0}{10} = 0$.

(6) $\lim_{x\to a} \frac{h(x)}{f(x)} = \cancel{\frac{10}{0}}$ DNE

(7) $\lim_{x\to a} \sqrt{g(x)} = \cancel{\sqrt{-8}}$ DNE. $\lim_{x\to a} \sqrt{h(x)} = \sqrt{10}$.

(9) $\lim_{x\to a} \frac{1}{g(x)-h(x)} = \frac{1}{(-8-10)} = \frac{-1}{18}$

Squeeze Theorem for limits

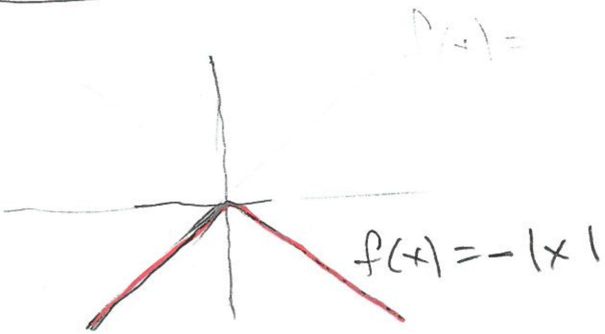
Suppose we have functions

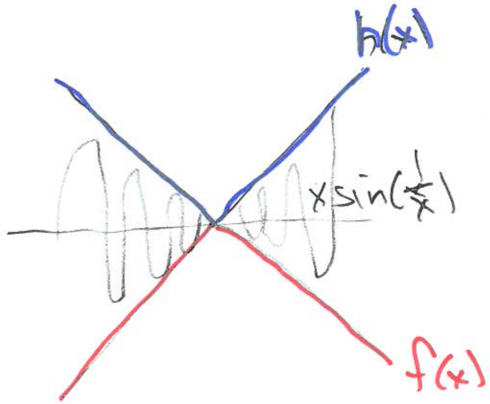
$$f(x) \leq g(x) \leq h(x)$$ (so g "squeezed" between the two).

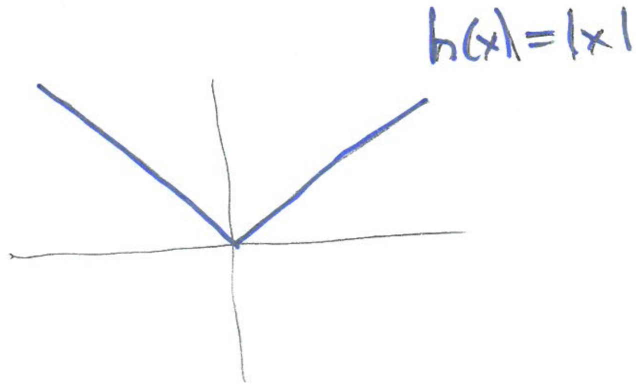
If $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} h(x) = L$,

then $\lim_{x \to a} g(x) = L$.

Example







$\lim_{x \to 0} -|x| = 0$

squeeze theorem says
because bigger and smaller
functions have limit 0 as $x \to 0$
the trapped function $x\sin\left(\frac{1}{x}\right)$ has limit 0 too.

$\lim_{x \to 0} |x| = 0$