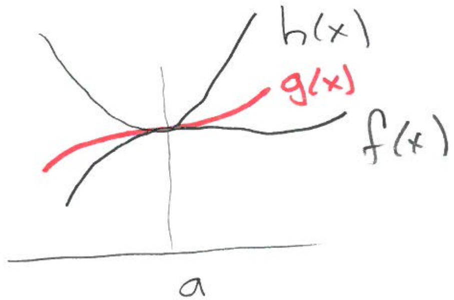
Squeeze Theorem for limits 3 functions

$f(x) \le g(x) \le h(x)$ g trapped between f and h

If $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} h(x) = L$ (same L).

then $\lim_{x \to a} g(x) = L$ too.



ww3 #12

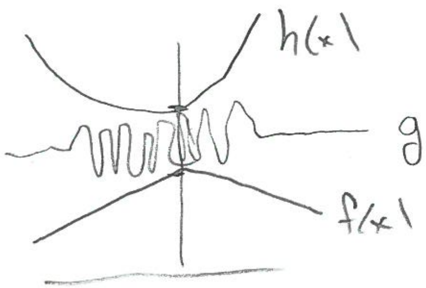
$(10x-26) \le f(x) \le (x^2+6x-22)$

$\lim_{x \to 2} (10x-26) = 10 \cdot 2 - 26 = -6$, $\lim_{x \to 2} (x^2+6x-22) = 2^2 + 6 \cdot 2 - 22$
$= 4 + 12 - 22 = -6$

same

Since f is trapped, $\lim_{x \to 2} f(x) = -6$.

Example



In this picture $\lim_{x \to a} h(x) = A$

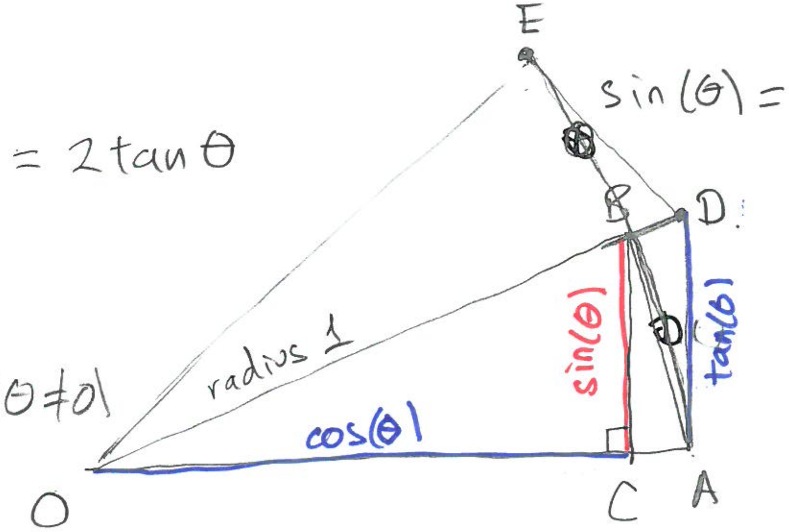
$\lim_{x \to a} f(x) = B$ $(B < A)$, $\lim_{x \to a} g(x)$ may not have limit

Very important trignometry limit.

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1.$$

| function $f(x) = \frac{\sin(\theta)}{\theta}$ does NOT allow input $\theta = 0$.

Use Squeeze Theorem to find limit.



$\sin(\theta) = BC < AB < \theta$

given $\frac{\sin(\theta)}{\theta} < 1 \quad (\theta \neq 0)$.

$2\theta < AD + DE = 2\tan\theta$

$\theta < \frac{\sin\theta}{\cos\theta}$

$\cos\theta < \frac{\sin\theta}{\theta} \quad (\theta \neq 0)$

Summary $\quad \cos\theta < \frac{\sin\theta}{\theta} < 1$

Since $\lim_{\theta \to 0} 1 = 1$ same.

$\lim_{\theta \to 0} \cos(\theta) = 1$

By squeeze theorem $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ too.

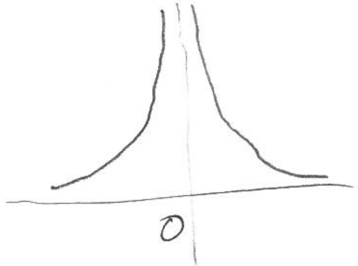
Will use fact $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ to find tangent slope of sine cosine.

Infinite limits

By $\lim_{x \to a} f(x) = +\infty$ we mean

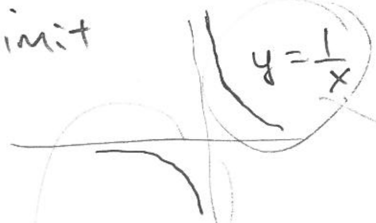
output values of f become large positive ("getting close to $+\infty$") when the input x becomes near to (but not equal to) a.

Examples ①



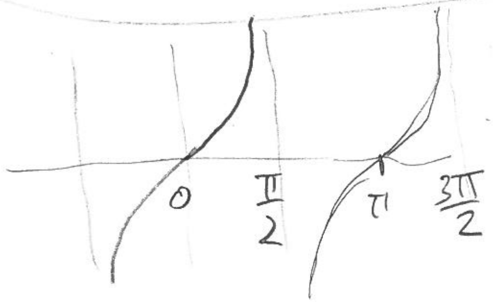
$f(x) = \frac{1}{x^2}$ $\quad \lim_{x \to 0} \frac{1}{x^2} = +\infty$.

② One sided limit



$\lim_{x \to 0^+} \frac{1}{x} = +\infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

③ $f(x) = \tan(x)$



$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty$

$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty$.

WW3 #14 Find $\lim\limits_{x \to -2^-}$, $\lim\limits_{x \to -2^+}$ of $f(x) = \dfrac{-3(x+2)}{x^2+4x+4}$

as $x \to -2$ $\dfrac{-3 \cdot 0}{4-8+4}$ $\dfrac{0}{0}$

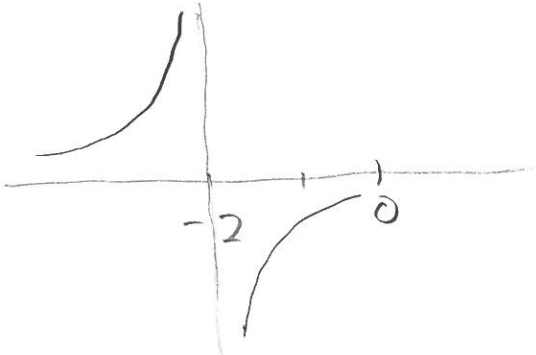
$x \neq -2$. $f(x) = \dfrac{-3(x+2)}{x^2+4x+4} = \dfrac{-3\cancel{(x+2)}}{(x+2)\cancel{(x+2)}} = \dfrac{-3}{(x+2)}$

f does NOT allow input -2

$\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} \dfrac{-3}{(x+2)} = +\infty$ $\left\{ \dfrac{-3}{(0^-)} \right.$

$\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} \dfrac{-3}{(x+2)} = -\infty$. $\left\{ \dfrac{-3}{0^+} \right.$

Since one-sided limits exists but are NOT equal $\lim\limits_{x \to 2} f(x)$ DNE.



We say the vertical line $x=-2$ is a <u>vertical asymptote</u>

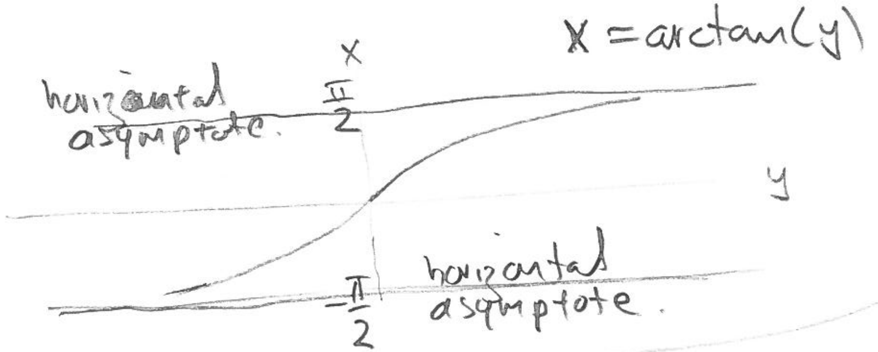
Vertical line $x=a$ is <u>vertical asymptote</u> for function $f(x)$, if either $\lim\limits_{x \to a^+} f(x) = +\infty$ or $-\infty$

OR $\lim\limits_{x \to a^-} f(x) = +\infty$, or $-\infty$

Limits at infinity: We say $\lim\limits_{x \to +\infty} f(x) = L$.

if the output values are close to L whenever the input x is large.

Example ① arctan



$$\lim_{y \to +\infty} \arctan(y) = \frac{\pi}{2}$$

$$\lim_{y \to -\infty} \arctan(y) = -\frac{\pi}{2}.$$

When $\lim\limits_{x \to \infty} f(x) = L$ or

$$\lim_{x \to -\infty} f(x) = L$$

then horizontal line $y = L$ is a <u>horizontal asymptote</u>

② WW3 #16 $f(x) = \left(\sqrt{x^2 - 9x + 1} - x\right)$

① Find $\lim\limits_{x \to +\infty} f(x)$.

$\lim\limits_{x \to +\infty} -x = -\infty$

$\lim\limits_{x \to +\infty} \sqrt{x^2 - 9x + 1} = \sqrt{\infty} = \infty$.

$\infty + \infty = \infty$

using "sum" rule $\infty - \infty$. does not have value.

Go back and work with $f(x)$. Try to manipulate it to get "simpler" expression.

$$f(x) = \left(\sqrt{x^2 - 9x + 1} - x\right) \cdot \left(\frac{\sqrt{x^2 - 9x + 1} + x}{\sqrt{x^2 - 9x + 1} + x}\right) = \frac{(\cancel{x^2} - 9x + 1) - \cancel{x^2}}{\sqrt{x^2 - 9x + 1} + x}$$

$$= \frac{-9x + 1}{\sqrt{x^2 - 9x + 1} + x} = \text{more manipulation} = \frac{\left(\frac{1}{x}\right)(-9x + 1)}{\left(\frac{1}{x}\right)\left(\sqrt{x^2 - 9x + 1} + x\right)} = \frac{-9 + \left(\frac{1}{x}\right)}{\left(\sqrt{1 - \frac{9}{x} + \frac{1}{x^2}} + 1\right)}$$

$\frac{-9 \cdot \infty}{\infty + \infty}$

$$\lim_{x \to +\infty} \frac{-9 + \left(\frac{1}{x}\right)}{\left(\sqrt{1 - \frac{9}{x} + \frac{1}{x^2}} + 1\right)} = \frac{-9 + 0}{\left(\sqrt{1 - 0 + 0} + 1\right)} = \frac{-9}{2}.$$

(2) $\lim_{x \to -\infty} \left(\sqrt{x^2 - 9x + 1} - x \right)$

$= +\infty + \infty = +\infty$.

$\lim_{x \to -\infty} (-x) = +\infty$

$\lim_{x \to -\infty} \sqrt{x^2 - 9x + 1} = +\infty$