Week 04 W L04 11am

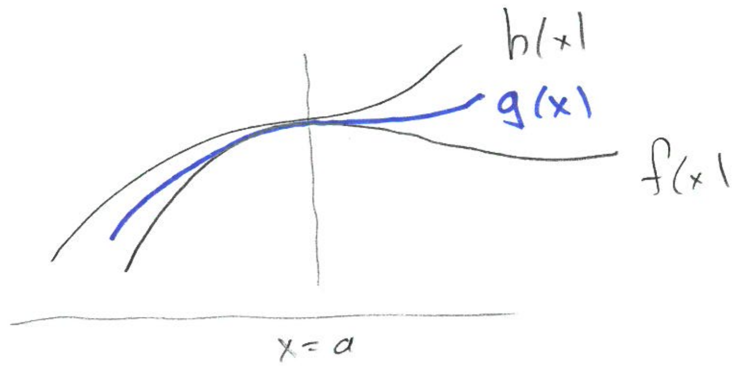
Squeeze Theorem If we have 3 functions with

$$f(x) \le g(x) \le h(x) \qquad g \text{ in between}$$

and $\lim\limits_{x \to a} f(x) = L$, and $\lim\limits_{x \to a} h(x) = L$ (SAME limit)

then the "trapped" function g also has $\lim\limits_{x \to a} g(x) = L$.

Picture



WW3 #12. function f trapped between

$$10x - 26 \le f(x) \le x^2 + 6x - 22.$$

What can we say about $\lim\limits_{x \to 2} f(x)$?

Since both larger/smaller functions have SAME lim -6 as $x \to 2$, the trapped function $f(x)$ does too.

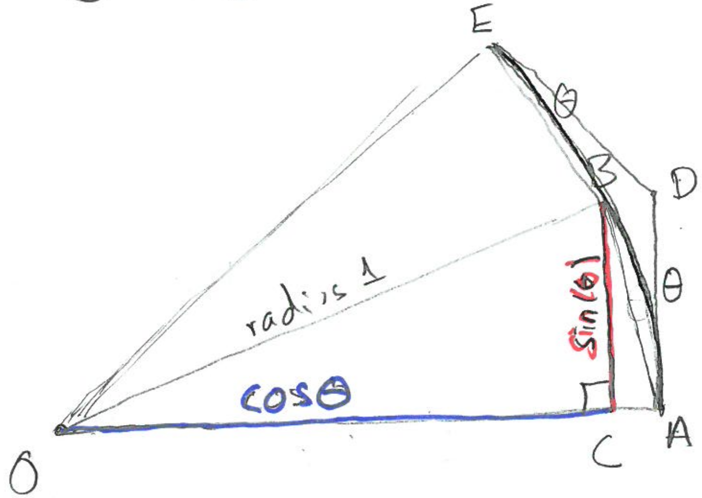
$\lim\limits_{x \to 2} (10x - 26) = 10 \cdot 2 - 26 = -6$

– and

$\lim\limits_{x \to 2} (x^2 + 6x - 22) = 2^2 + 6 \cdot 2 - 22 = -6$

Use of squeeze theorem to prove important limit

We show $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$. | Note. $f(\theta) = \frac{\sin(\theta)}{\theta}$ does NOT allow input $\theta = 0$



$\sin(\theta) = BC < BA < \theta$.

so $\frac{\sin(\theta)}{\theta} < 1$. $(\theta \neq 0)$.

$2\theta < AD + DE = 2\tan\theta = 2\frac{\sin\theta}{\cos\theta}$ gives $\cos\theta < \frac{\sin\theta}{\theta} < 1$

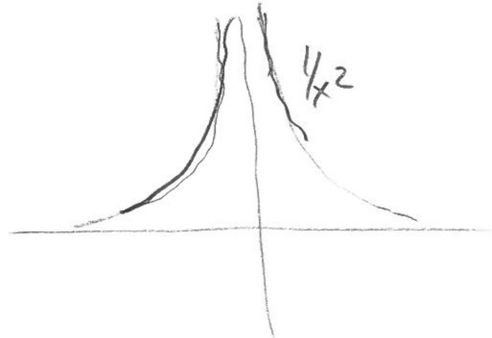
By our pictures we have trapped function $\frac{\sin\theta}{\theta}$ $(\theta \neq 0)$ between $\cos\theta$ and $\underline{1}$. Since $\lim_{\theta \to 0} 1 = 1$, $\lim_{\theta \to 0} \cos\theta = 1$ $(1 = 1)$

So squeeze theorem say $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

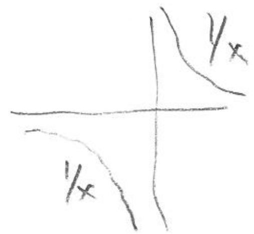
Infinite limits

$\lim_{x \to a} f(x) = +\infty$ if the output values $f(x)$ are large positive whenever x is near to (but not equal to) a

Examples ① $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.



② $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.



③ $\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty$

$\lim_{x \to \frac{\pi}{2}^+} \tan(x) = -\infty$

one-sided (NON equal) limits

$\lim_{x \to \frac{\pi}{2}} \tan(x) = \text{DNE}$.

Remark. If

$\lim_{x \to a^+} f(x) = +\infty$ or $-\infty$

or

$\lim_{x \to a^-} f(x) = +\infty$ or $-\infty$

we say vertical line $x = a$ is vertical asymptote

WW3 #14

does NOT allow input $x=-2$

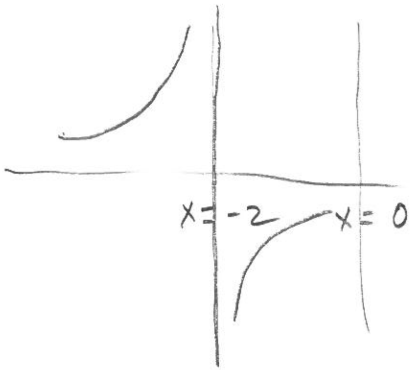
(1) $\lim_{x \to -2^-} \dfrac{-3(x+2)}{x^2+4x+4} = \lim_{x \to -2^-} \dfrac{-3\cancel{(x+2)}}{(x+2)\cancel{(x+2)}} = \lim_{x \to -2^-} \dfrac{-3}{(x+2)}$. $\dfrac{-3}{0^-}$

$= +\infty$

(2) $\lim_{x \to -2^+} \dfrac{-3(x+2)}{x^2+4x+4} = \cdots = \lim_{x \to -2^+} \dfrac{-3}{(x+2)}$ $\quad \dfrac{-3}{0^+}$

$= -\infty$

Since one-sided limits are NOT equal, $\lim_{x \to -2} \dfrac{-3(x+2)}{x^2+4x+4} = \text{DNE}$



vertical line $x=-2$ is vertical asymptote.

Limits at infinity

We say $\lim\limits_{x \to +\infty} f(x) = L$ means that output values $f(x)$ are near L whenever the input x is large positive.

Example. ①

$$\lim_{x \to +\infty} \left(\frac{1}{x}\right) = 0.$$

WW3 # 16 Find

$\infty + \infty = \infty$

$\infty - \infty = ?$

(1) $\lim\limits_{x \to +\infty} \left(\sqrt{x^2 - 9x + 1} - x\right).$

$\cancel{\frac{-9\infty}{\infty}}$

Need to manipulate function

$$\left(\sqrt{x^2 - 9x + 1} - x\right) \cdot \frac{\left(\sqrt{x^2 - 9x + 1} + x\right)}{\left(\sqrt{x^2 - 9x + 1} + x\right)} = \frac{(\cancel{x^2} - 9x + 1) - \cancel{x^2}}{\left(\sqrt{x^2 - 9x + 1} + x\right)} = \frac{(-9x + 1)}{\left(\sqrt{x^2 - 9x + 1} + x\right)}$$

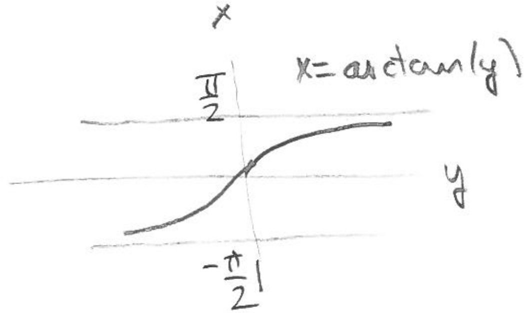
$$= \frac{\left(\frac{1}{x}\right)(-9x + 1)}{\left(\frac{1}{x}\right)\left(\sqrt{x^2 - 9x + 1} + x\right)} = \frac{(-9 + 1/x)}{\left(\sqrt{1 - \frac{9}{x} + \frac{1}{x^2}} + 1\right)}$$

But $\lim\limits_{x \to +\infty} \dfrac{(-9 + 1/x)}{\left(\sqrt{1 - \frac{9}{x} + \frac{1}{x^2}} + 1\right)} = \dfrac{-9 + 0}{\left(\sqrt{1 + 0 + 0} + 1\right)}$

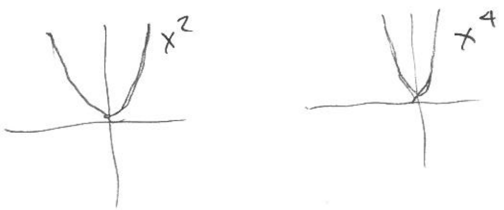
$= \dfrac{-9}{2}.$

(2) $\lim_{x \to -\infty} \left(\sqrt{x^2 - 9x + 1} - x\right) = (+\infty - (-\infty)) = +\infty + \infty = +\infty$

WW3 #17 Find $\lim_{x \to +\infty} \arctan(x^2 - x^4)$.



What happens to $(x^2 - x^4)$ as $x \to +\infty$.



shape similar but x^4 grows much more rapids.

Ex At input $x = 10^6$, $x^2 = 10^{12}$
$x^4 = 10^{24}$

Both x^2, x^4 grow towards $+\infty$ as $x \to +\infty$,

But $(x^2 - x^4) \to -\infty$ as $x \to +\infty$.

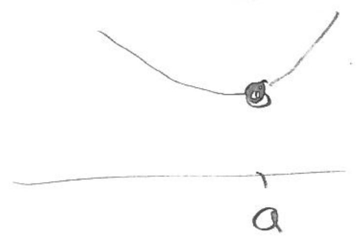
$$\lim_{x \to +\infty} \arctan(x^2 - x^4) = \lim_{y \to -\infty} \arctan(y) = \frac{-\pi}{2}.$$

Reminder. When we talk about $\lim_{x \to a} f(x)$, we

DO NOT care whether a is an input for f, and even if a is allowed input, we do not care about f(a).

For most comman functions, f when we take $\lim_{x \to a} f(x)$,

a is an allowed input of f.



When $\lim_{x \to a} f(x) = L$ equals value f(a).

we say function is continuous at input a