Horizontal asymptote If $\lim_{x \to \infty} f(x) = L$, we say

$f(x)$ has horizontal line $y = L$ as horizontal asymptote

Example ① WW3 #16. $\lim_{x \to \infty} \left(\sqrt{x^2 - 9x + 1} - x\right) = -\frac{9}{2}$.

Can think of $y = -\frac{9}{2}$ is a constant function

$$\lim_{x \to \infty} \left(\left(\underbrace{\sqrt{x^2 - 9x + 1}}_{\text{1st function}} - x \right) - \underbrace{\left(-\tfrac{9}{2}\right)}_{\text{constant function}} \right) = 0$$

If $f(x), g(x)$ are two functions $\lim_{x \to \infty} (f(x) - g(x)) = 0$

we say f, g are asymptotic.

$f(x) = \sqrt{x^2 - 9x + 1}$, $g(x) = \left(x + \left(-\frac{9}{2}\right)\right)$ line function

We have $\lim_{x \to \infty} \left(\underbrace{\sqrt{x^2 - 9x + 1}}_{\text{new 1st function}} - \underbrace{\left(x + \left(-\frac{9}{2}\right)\right)}_{\text{new 2nd function}} \right) = 0$

We say the line $y = x + \left(-\frac{9}{2}\right)$ is slant asymptote of $\sqrt{x^2 - 9x + 1}$

Growth of functions If f, g are 2 functions and both $\lim_{x \to +\infty} f(x) = \infty$, $\lim_{x \to \infty} g(x) = \infty$, we say

f, g grow at relatively same rate if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \ (\neq 0)$.

and g grow faster if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$

Example ① $f(x) = \sqrt{x^2 - 9x + 1}$
$g(x) = x$

We have
$$\frac{f(x)}{g(x)} = \frac{\sqrt{x^2-9x+1}}{x} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{\sqrt{1 - \frac{9}{x} + \frac{1}{x^2}}}{1} \longrightarrow \frac{\sqrt{1+0+0}}{1}$$
as $x \to \infty$.

So f, g have same relative rate of growth

② Parabola $g(x) = x^2$, line $f(x) = 9900x$.

Ratio
$$\frac{f(x)}{g(x)} = \frac{9900x}{x^2} = \frac{9900}{x} \longrightarrow 0 \text{ as } x \to \infty.$$

So x^2 grows faster than x as $x \to \infty$.
parabola function grows faster than linear function.

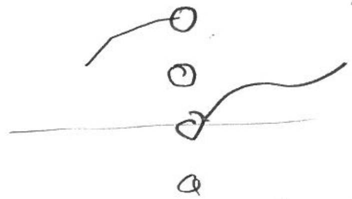
More generally $f(x) = x^m + a_1 x^{m-1} + \dots + a_m$ and $m < n$

$g(x) = x^n + b_1 x^{n-1} + \dots + a_n$

degree g > degree f, then $\lim\limits_{x \to \infty} \frac{f(x)}{g(x)} = 0$, so

polynomial with larger degree grows faster

Continuity Whether limit $\lim\limits_{x \to a} f(x)$ exists or does not exist has NOTHING to do with value of f at a. (In fact a may not be in domain).



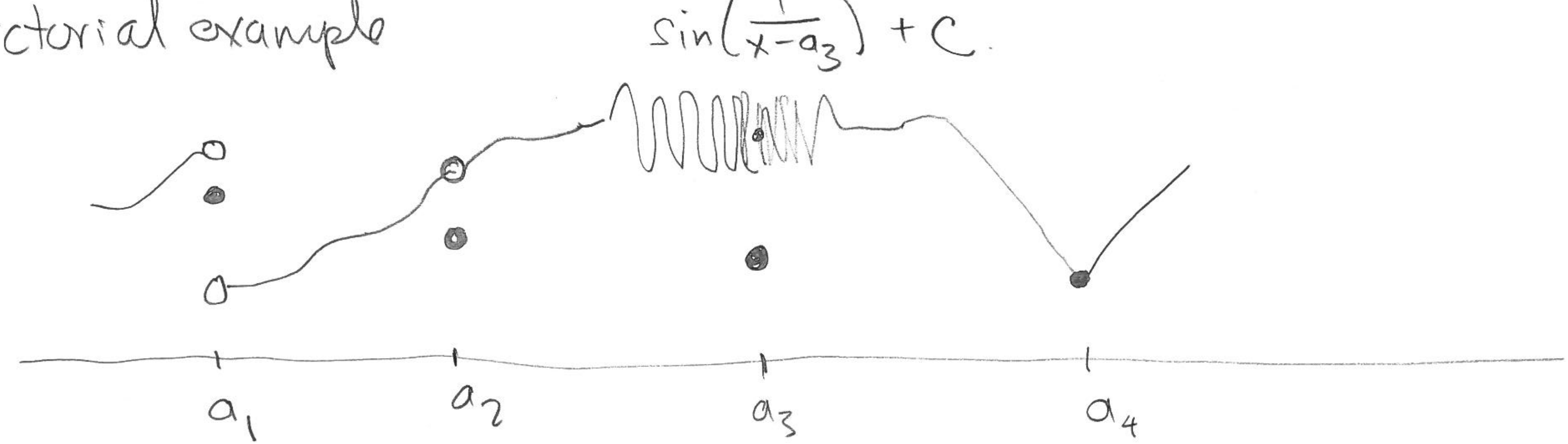
For many common functions f on an interval I we have

① $\lim\limits_{x \to a} f(x) = L$ exists

② a is in domain of f and value $f(a)$ equal L.

When this happens we say f is continuous at input a.

Pictorial example



NOT continuous

a_1 : left, right side limit exists, but are not equal so $\lim_{x \to a_1} f(x) = \text{DNE}$.

a_1 is in domain, but $f(a_1)$ is NOT equal to either left or right one-side limits.

a_2 left/right one-sided limits exists and equal, but NOT equal to $f(a_2)$.

a_3 neither one-side limit exists, $f(a_3)$.

continuous

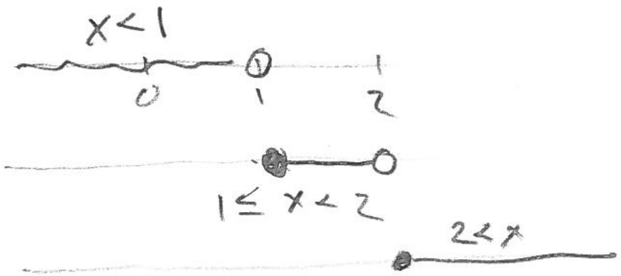
a_4 left/right one sided limits exists, and are equal and equal to $f(a_4)$

ALL OTHER inputs a, f is continuous.

We say a function is <u>continuous everywhere</u> if it is continuous at all points in its domain

<u>Intuition</u>. f is continuous if we can draw its graph without lift our pencile off paper

WW3 #18 $f(x) = \begin{cases} 2x \\ cx^2+d \\ 5x \end{cases}$



Find c, d so function continuous

Polynomials are continuous functions

@ $a=1$ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2\cdot 1 = 2 = f(1)$ if continuous
$= c\,1^2+d$

so $\left.\begin{aligned} 2 &= c+d \\ 10 &= 4c+d \end{aligned}\right\}$ solve

@ $a=2$ $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} cx^2+d = c\cdot 2^2+d = 4c+d$
$= f(2) = 5\cdot 2 = 10$

$8 = 3c$

$\boxed{\begin{aligned} c &= 8/3 \\ d &= -2/3 \end{aligned}}$

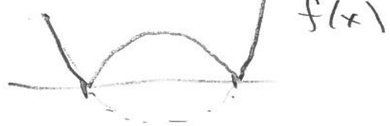
Derivative (fancy name for tangent slope)

Setting. f is a function on an interval I

$a \in I$. If

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L \text{ exists}$$ (geometric meaning limit of secant slopes = tangent slope at a)

we say f is differentiable at input a, and the value is L.



Example $f(x) = \left| \frac{1}{4}x^2 - x \right|$

$$= \begin{cases} \frac{1}{4}x^2 - x & \text{for } x < 0 \text{ OR } 4 < x \\ -\left(\frac{1}{4}x^2 - x\right) & \text{for } 0 \le x \le 4 \end{cases}$$

We saw

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \begin{cases} \frac{1}{2}a-1 & a<0 \text{ OR } 4<a. \\ -\frac{1}{2}a+1 & 0<a<4 \\ \text{DNE} & a=0, \ a=4 \end{cases}$$

Example $f(x) = x^n$ n positive integer 1, 2, 3, ...

$$\frac{f(x)-f(a)}{x-a} = \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2} \cdot a + x^{n-3} \cdot a^2 + \cdots + x^1 \cdot a^{n-2} + a^{n-1}$$

$(x \neq a)$.

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \left(x^{n-1} + x^{n-2} a^1 + x^{n-3} a^2 + \cdots + x^1 \cdot a^{n-2} + a^{n-1} \right)$$

$$= a^{n-1} + a^{n-2} \cdot a + a^{n-3} a^2 + \cdots + a \cdot a^{n-2} + a^{n-1}$$

$= n \cdot a^{n-1}$ exists. Derivative of $f(x) = x^n$ at input a is na^{n-1}.

$x \to a$ as $x \to a$