Derivative as instantaneous rate of change

Suppose $p(t)$ = position of point at time t.

If $a < t$, then

$$\frac{p(t)-p(a)}{t-a} = \frac{\text{amount point has moved}}{\text{time}}$$

= average speed during interval $[a, t]$.

$$\lim_{t\to a} \frac{p(t)-p(a)}{t-a} = \text{instantaneous speed at time } a.$$

More generally if Q is a quantity dependent on variable s, then

$$\lim_{s\to a} \frac{Q(s)-Q(a)}{s-a}$$ = (when it exists) is instantaneous rate of change of Q with respect to s.

Example ① Area of circular disk in terms of radius.

$$A(r) = \pi r^2$$

$$\lim_{r\to a} \frac{\pi r^2 - \pi a^2}{r-a} = \lim_{r\to a} \frac{\pi(r+a)\cancel{(r-a)}}{\cancel{(r-a)}} = \pi(a+a) = 2\pi a.$$

increase radius by Δa, area increases by $\underline{(2\pi a)\Delta a}$

For Δa change in radius, area increases by $(2\pi\, 2a)\Delta a$

(2) Formula to change temperature from $F°$ to $C°$

$$C(F) = \frac{5}{9}(F-32)$$

$$\lim_{F \to a} \frac{C(F)-C(a)}{F-a} = \lim_{F \to a} \frac{\frac{5}{9}(F-32) - \frac{5}{9}(a-32)}{F-a} = \frac{5}{9}.$$

That instantaneous rate of change is constant $\left(\frac{5}{9}\right)$ for each ΔF change in F, C changes by $\Delta C = \left(\frac{5}{9}\right)\Delta F$

Derivative at single input VS Derivative function.

Single input: $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L$ (when it exists).

value of limit is called value of derivative for input a

Derivative function Function whose domain $\{a \mid \text{derivative at input } a \text{ exists}\}$

Use $a \xrightarrow{f'}$ value of derivative at input a

$f'(a) =$ limit

Example $f(x) = x^{1/3}$ cube root function. (domain of f is $\mathbb{R}$).

$$\frac{f(x)-f(a)}{x-a} = \frac{x^{1/3}-a^{1/3}}{x-a} \cdot \frac{(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}{(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$(r-s)\cdot(r^2+rs+s^2) = (r^3-s^3)$

$r = x^{1/3}$, $s = a^{1/3}$

$$= \frac{x-a}{x-a} \frac{1}{(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})}$$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{1}{(x^{2/3}+x^{1/3}a^{1/3}+a^{2/3})} = \frac{1}{(a^{2/3}+a^{1/3}a^{1/3}+a^{2/3})}$$

$$= \frac{1}{3a^{2/3}} = \frac{1}{3}a^{-2/3}.$$

$(a \neq 0)$. Derivative at input a.

Derivative function $f'(a) = \frac{1}{3}a^{-2/3}$ $(a \neq 0)$.

Basic derivative. $f(x) = x^n$ n positive integer

$$\frac{f(x)-f(a)}{x-a} = \frac{x^n - a^n}{x-a}$$

$$\begin{array}{r l}
 & x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots \quad xa^{n-2} + a^{n-1} \\
x-a \,\big) & x^n \qquad\qquad\qquad\qquad\qquad\qquad -a^n \\
 & x^n - ax^{n-1} \\
\hline
 & ax^{n-1} \\
 & ax^{n-1} + a^2x^{n-2} \\
\hline
 & a^2x^{n-2}
\end{array}$$

$$= x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}$$

n terms.

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \left(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}\right) = a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1}$$

$= na^{n-1}$ derivative at input a.

Derivative function $f'(a) = na^{n-1}$. | Actually true for n equal to any number.

$f(x) = x^{-1}$ $(n=-1)$

$f'(a) = (-1)a^{-1-1} = (-1)a^{-2}$

<u>Basic derivate rules</u>: Suppose f, g are two functions with derivatives

$$(f \pm g)'(a) = f'(a) \pm g'(a)$$ sum/difference rule.

If c is a constant multiplication by constant.

$$(c \cdot f)'(a) = c\, f'(a)$$

Example. We can find derivative of any polynomial.

$f(x) = x^3 - 2x^2 + 1$, has $f'(a) = 3a^2 - 2 \cdot (2a^1) + 0$

$$f'(a) = 3a^2 - 4a$$

WW4

#5 $(\text{red})' = \text{green}$ $(\text{blue})' = \text{red}$

blue $= f$, red $= f'$, green $= f''$

#7 Find $f'(z)$ for $f(z) = \dfrac{z^7 + 7}{z^{1/2}} = z^{6\frac{1}{2}} + 7 \cdot z^{-1/2}$ | $f'(z) = (6\tfrac{1}{2}) z^{5\frac{1}{2}} + 7 \cdot (-\tfrac{1}{2}) z^{-3/2}$