Week 5 F 204 11am

Midterm Sunday 28 Oct.

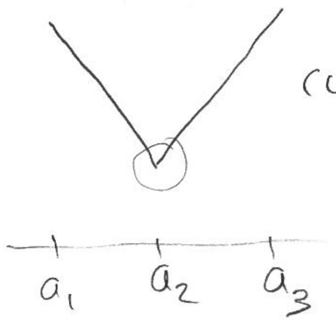
Information page link at course webpage.

Sample midterm from previous years.

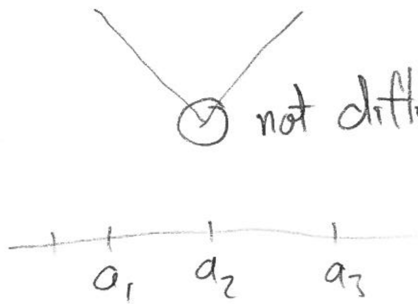
Continuous VS Differentiable.

For both the approach point a needs to be in domain

$$\lim_{x \to a} f(x) = f(a) \qquad\qquad \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = L \text{ exists.}$$



continuous at a_2



not differentiable at a_2.

continuous $\not\Rightarrow$ differentiable

But differentiable $\Longrightarrow$ continuous (at input a).

Continuous is $\lim_{x \to a} f(x) = f(a)$ SAME $\lim_{x \to a} (f(x) - f(a)) = 0$.

Now if $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$ exists, we use fact

$\lim_{x \to a} (x - a) = 0$. $\longrightarrow$ combine these two facts

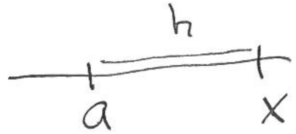
$$\left(\frac{f(x) - f(a)}{x - a}\right) \longrightarrow L$$

$$(x - a) \longrightarrow 0$$

By product rule $\left(\dfrac{f(x) - f(a)}{x - a}\right) \cdot (x - a) \longrightarrow L \cdot 0 = 0$

thus $(f(x) - f(a)) \longrightarrow 0$

Derivative:
$$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x-a} \right) = L \text{ exists.}$$



$x = a+h$

$h = x-a$.

SAME AS
$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right) = L \text{ exists}$$

Derivative of trig functions sin and cos.

$$\frac{\sin(a+h) - \sin(a)}{h}$$ does this have limit?

By trig identity
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(a+h) = \sin(a)\cos(h) + \cos(a)\sin(h)$$

$$\frac{\sin(a+h) - \sin(a)}{h} = \frac{\sin(a)\cos(h) - \sin(a) \quad + \cos(a)\sin(h)}{h}$$

$$\frac{\sin(a+h)-\sin(a)}{h} = \cos(a)\left(\frac{\sin(h)}{h}\right) + \sin(a)\left(\frac{\cos(h)-1}{h}\right)$$

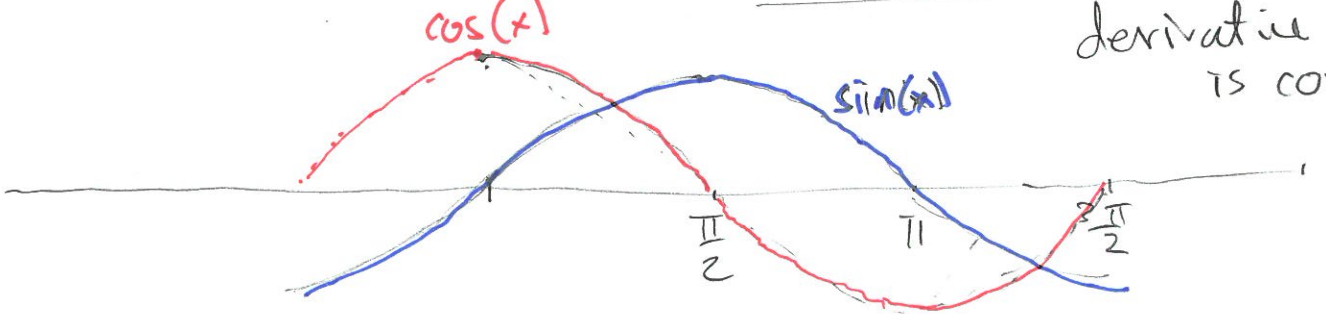
As $h \to 0$, recall squeeze theorem $\left(\frac{\sin h}{h}\right) \to 1$.

also by squeeze theorem $\left(\frac{\cos(h)-1}{h}\right) \to 0$.

So $\lim\limits_{h\to 0} \frac{\sin(a+h)-\sin(a)}{h} = \cos(a)\cdot 1 + \sin(a)\cdot 0 = \cos(a)$.

Derivative function of sin is $\underline{(\sin'(a)) = \cos(a)}$.

derivative of sine is cosine



For cosine we need to find

$$\lim_{h \to 0} \frac{\cos(a+h) - \cos(a)}{h} =$$ use trig identity for $\cos(A+B)$ and same two limits $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$

$$\lim_{h \to 0} \frac{1-\cos(h)}{h} = 0$$

$$= -\sin(a)$$

Derivative of cosine is $-$sine.

$(\cos'(a)) = -\sin(a)$

Two more important differentiation rules f, g differentiable

PRODUCT $(fg)' = f'g + f \cdot g'$

QUOTIENT Assume $g(a) \neq 0$. $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$

WW4 #8 Find derivative of $f(x) = \dfrac{12\sin(x) - 4}{\cos(x)}$

Use quotient rule:

$$\left(\frac{12\sin(x)-4}{\cos(x)}\right)' = \frac{(12\sin(x)-4)'\cos(x) - (12\sin(x)-4)(\cos(x))'}{(\cos(x))^2}$$

$$= \frac{12\cos(x)\cos(x) - (12\sin(x)-4)(-\sin(x))}{(\quad)^2}$$

$$= \frac{12\left((\cos(x))^2 + (\sin(x))^2\right) - 4\sin(x)}{(\cos x)^2} = \frac{12 - \sin(x)}{(\cos(x))^2}$$

#6. Suppose $f(-5) = 5 \quad f'(-5) = -5$

$g(-5) = -4, \quad g'(-5) = 4$

Compute $(fg)'(-5)$. $\quad (fg)'(-5) = f'(-5)g(-5) + f(-5)g'(-5)$

$$= (-5)(-4) + 5 \cdot 4 = 40.$$

Derivative of composition of two functions

f, g differentiable functions, and $f \circ g$ makes sense

COMPOSITION RULE Derivative at a of $f \circ g$ | $(f \circ g)(a) = f(g(a))$

CHAIN RULE

$$(f \circ g)'(a) = f'(g(a))\, g'(a)$$

Why true?

$$\frac{f(g(a+h)) - f(g(a))}{h} = \frac{f(y) - f(b)}{(y-b)} \frac{(y-b)}{h}$$

New variables
$y = g(a+h)$
$b = g(a)$

As $h \to 0$, what happens?

We have $\frac{y-b}{h} = \frac{g(a+h) - g(a)}{h} \to g'(a)$

As $h \to 0$, the new variable $y = g(a+h) \longrightarrow b$ so $\lim_{y \to b} \left(\frac{f(y) - f(b)}{y-b} \right) = f'(b)$

So $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

WW4 # 9 Find derivative of $w(r) = \sqrt{r^7+5}$

Function w is composite of $\sqrt{}$ outside f

r^7+5 inside g.

$g(r) = r^7+5$ has derivative $g'(r) = 7r^6 + 0 = 7r^6$.

$f(s) = s^{1/2}$ has derivative $f'(s) = \frac{1}{2} s^{-1/2}$

So $w'(r) = (f \circ g)'(r) = f'(g(r))\ g'(r)$

$= \frac{1}{2}(r^7+5)^{-1/2}\ (7r^6)$

$\sqrt{}$ outside

#12 Find derivative of $f(x) = \sqrt{2 + (\sin(x))^2}$ $2+(\sin(x))^2$ inside

$f'(x) = \frac{1}{2}(2+(\sin(x))^2)^{-1/2} \cdot (0 + (\cos x)(\sin x) + (\sin x)(\cos x))$

$= \frac{1}{2}(2+(\sin(x))^2)^{-1/2} \cdot 2(\cos x)(\sin x)$.

#13 Given $\frac{d}{dx}(f(3x^2)) = 8x^4$ find $f'(x)$

outside function f
inside function $g(x) = 3x^2$

$$\frac{d}{dx}(f(3x^2)) = \frac{d}{dx}(f(g(x)))$$

$$= f'(g(x)) \cdot g'(x) \quad \text{(CHAIN RULE)}$$

$$= f'(3x^2) \cdot 3 \cdot 2x = f'(3x^2) \cdot 6x$$

Hypothesis is $f'(3x^2) \cdot 6x = \frac{d}{dx}(f(3x^2)) = 8x^4$

$$f'(3x^2) = \left(\frac{4}{3}\right)x^3.$$

Take $u = 3x^2$, inverse is $\sqrt{\frac{u}{3}} = x$, Replace x by $\sqrt{\frac{u}{3}}$

$$f'\left(3 \cdot \left(\sqrt{\tfrac{u}{3}}\right)^2\right) = \left(\frac{4}{3}\right)\left(\frac{u}{3}\right)^{3/2}$$

$$f'(u) = \left(\frac{4}{3}\right)\left(\frac{u}{3}\right)^{3/2}. \quad \Bigg| \quad f'(x) = \left(\frac{4}{3}\right)\left(\frac{x}{3}\right)^{3/2}$$