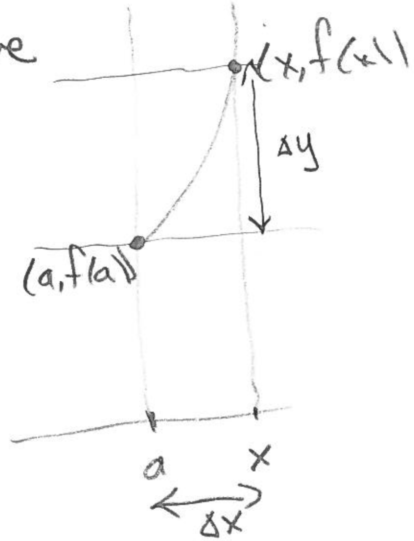
Leibniz notation for derivative

$f(x)$ function

$f'(x)$ derivative



$$y = f(x).$$

Recall definition of derivative / tangents

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$\Delta y = f(x) - f(a)$

$\Delta x = x - a$

Leibniz Think of infinitesimal change dx in x yields an infinitesimal change dy in y

tangent slope is $\dfrac{dy}{dx}$.

Chain rule Find derivative of $f(g(x))$

$y = g(x)$ inside function

$u = f(y)$ outside function

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$$

$$f'(g(x)) \quad g'(x)$$

The idea of infinitesimal dx will appear again in integral $\int f(x)\,dx$

Basic derivatives

(1) $\frac{d}{dx}(x^r) = r x^{r-1}$

(2) $\frac{d(\sin x)}{dx} = \cos x$

$\frac{d(\cos x)}{dx} = -\sin x$

Basic derivative rules

SUM/DIFFERENCE

PRODUCT $(fg)' = f'g + fg'$

QUOTIENT

CHAIN

Derivative of exponential $y = f(x) = b^x$

$b^{a+h} = b^a b^h$

We will show derivative exists and answer relate to tangent slope at point $(0,1)$

$b^a = b^{a+0} = b^a b^0$

$$\frac{f(a+h) - f(a)}{h} = \frac{b^{a+h} - b^a}{h} = \frac{b^a b^h - b^a b^0}{h} = b^a\left(\frac{b^h - b^0}{h}\right)$$

Now image $h \to 0$, $\lim_{h\to 0} b^a\left(\frac{b^h - b^0}{h}\right) = b^a \lim_{h\to 0}\left(\frac{b^h - b^0}{h}\right)$

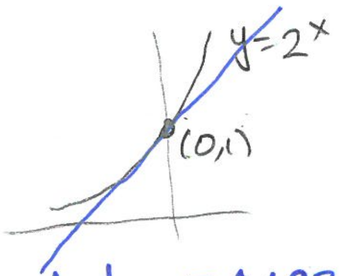
derivative tangent slope at point $(0,1)$

So derivative of $f(x) = b^x$ is

$f'(a) = b^a \cdot$ tangent slope at $(0,1)$

Experimental/numerical examples.

① $y = 2^x$



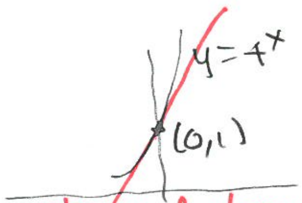
tangent slope = 0.6931...

h	$\frac{2^h - 1}{h}$
0.01	0.6956...
0.001	0.6933...
0.0001	0.6931...
0.00001	0.6931...

tangent slope to graph at point $(0,1)$ is $0.6931..$

$$(2^x)' = 2^x \cdot (0.6931\ldots).$$

② $y = 4^x = 2^{2x}$



Graph of $y = 4^x$ is contracted in horizontal direction by a factor of 2

tangent slope = 1.3863...

tangent slope of $y = 4^x$ at point $(0,1)$ is $2 \cdot (0.6931) = 1.3863\ldots$

$$(4^x)' = 4^x \cdot (1.3863\ldots).$$

For $y=2^x$ tangent slope at $(0,1)$ is <1
$y=4^x$ ——————————— >1

There is a remarkable number e between 2 and 4 so that $y=e^x$ has tangent slope $\underline{1}$ at point $(0,1)$.

For this remarkable number e

$$(e^x)' = e^x \cdot 1$$

For base e, the logarithm $\log_e$ is abbreviated as ln
$\log_e(y) = \ln(y)$

The number e is $2.718281828\ldots$

We can use fact $(e^x)'=e^x$ to find other derivatives $y=b^x$

$$b^x = (e^{\log_e(b)})^x = e^{\log_e(b)\cdot x} \qquad e^{\log_e(b)} = b$$

we have written b^x as composition $\underset{\text{outside function}}{e^u}$ and $\underset{\text{inside function}}{u = \log_e(b)x}$

So by chain rule

$$(b^x)' = e^u \cdot \log_e(b) = b^x \cdot \boxed{\log_e(b)}$$

This must be tangent slope to b^x at point $(0,1)$.

So tangent slope at $(0,1)$ of graph of b^x is $\underline{\underline{\ln(b)}}$

2^x tangent slope at $(0,1)$ is $\ln(2) = 0.6931\ldots$

4^x ——————— $\ln(4) = 1.3862\ldots$

10^x ——————— $\ln(10) = 2.302\ldots$

WW4 #10 Find derivative of $y = \dfrac{e^{5x}}{x^7+1}$ quotient of $\dfrac{e^{5x}}{x^7+1}$

To find derivative of e^{5x}, we note $e^{5x} = e^u$, $u = 5x$

outside inside

so $(e^{5x})' = e^u \cdot 5 = e^{5x} \cdot 5$

By quotient rule

$$\left(\frac{e^{5x}}{x^7+1}\right)' = \frac{(e^{5x}\cdot 5)(x^7+1) - (e^{5x})\cdot(7x^6+0)}{(x^7+1)^2}.$$

#11 Find derivative of $w = (t^2+5t).(1-e^{-2t})$

Use product rule, and to find derivative of e^{-2t}, we treat it as composition e^u, $u=-2t$

So $(e^{-2t})' = e^u \cdot (-2) = e^{-2t}(-2)$.

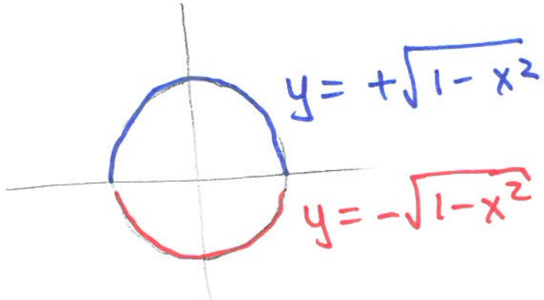
Then by product rule

$$\frac{dw}{dt} = (2t+5)(1-e^{-2t}) + (t^2+5t)(0 - e^{-2t}(-2))$$

Implicit Functions and Implicit Differentiation

Sometimes we only have a relation among variables, and not an explicit formula for one variable in terms of other.

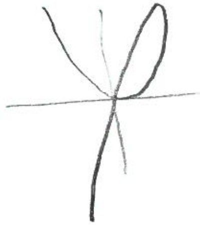
Examples. ① $x^2+y^2=1$



This relation is very simple.

Here we can solve for y in terms of x $y^2=1-x^2$, $y=\sqrt{1-x^2}$, $y=-\sqrt{1-x^2}$

② $x^3+y^2=6xy$. has graph



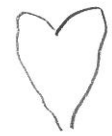
see link at course webpage

Here since only y^2, $(6x)y$ appear, we can use quadratic formula to get formula for y in terms of x

③ $x^3 + y^3 = 3xy^2 - x - 1$

Here difficult to solve for y in terms of x or x in terms of y.

④ $(x^2 + y^2 - 1)^3 - x^3 y^3 = 0$ heart shaped.



Again difficult to solve for x or y in terms of other.

⑤ WW5 #1 $x^5 + 4xy + y^4 = 41$ has point $(2,1)$

Verify $(2^5) + 4 \cdot 2 \cdot 1 + (1)^4 \stackrel{?}{=} 41$

$32 + 8 + 1 \stackrel{\checkmark}{=} 41$ ✓

Find tangent slope and tangent line at $(2,1)$

implicit differentiation