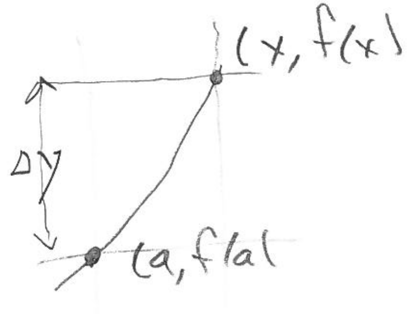
Alternative (better) notation for derivative

Invented by Leibniz. $y = f(x)$ $\frac{dy}{dx}$ is derivative.

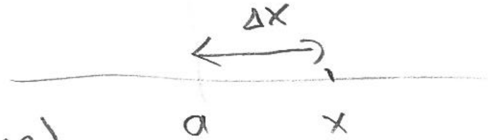
From definition

$$\frac{f(x)-f(a)}{x-a} = \frac{\Delta y}{\Delta x}$$

$\Delta y = f(x) - f(a)$
$\Delta x = x - a$.



As $x \to a$, the quantity $\Delta x \to 0$
$\Delta y \to 0$



ratio $\left(\frac{\Delta y}{\Delta x}\right)$ has limit. (tangent slope).

$$\frac{\Delta y}{\Delta x} \longrightarrow \frac{dy}{dx}$$

Derivative is ratio of 2 infinitesimals $\frac{dy}{dx}$

Integrals $\int \_ \, dx$

Chain Rule

$f(g(x))$ has derivative

$y = g(x)$ $u = f(y)$

$$\frac{d(f \circ g)}{dx} = \underbrace{\frac{df}{dy}}_{f'(g(x))} \cdot \underbrace{\frac{dy}{dx}}_{g'(x)}$$

Basic derivatives

(1) $$\frac{d(x^r)}{dx} = r x^{r-1}$$

(2) $$\frac{d(\sin x)}{dx} = \cos x, \quad \frac{d(\cos x)}{dx} = -\sin x$$

Basic rules

(1) SUM / DIFFERENCE

(2) PRODUCT

(3) QUOTIENT

(4) CHAIN.

"Proof" of Product rule

$$\lim_{h \to 0} \frac{f(a+h)\,g(a+h) - f(a)g(a)}{h}$$

$$\frac{f(a+h)g(a+h) - f(a)\,g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h}\right) g(a+h) + f(a)\left(\frac{g(a+h) - g(a)}{h}\right)$$

Because g is continuous

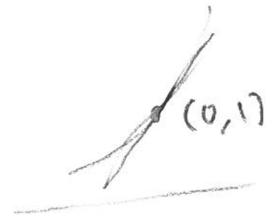
$$f'(a)\,g(a) + f(a)\,g'(a)$$

Derivative of exponential. $b > 1$

$$f(x) = b^x \qquad b^{a+h} = b^a \cdot b^h$$

$$\frac{f(a+h) - f(a)}{h} = \frac{b^{a+h} - b^a}{h} = \frac{b^a b^h - b^a}{h} = b^a \left(\frac{b^h - 1}{h} \right)$$

$$\lim_{h \to 0} \left(\frac{b^h - 1}{h} \right) = \text{tangent slope at } (0,1).$$



$$\left. \frac{d(b^x)}{dx} \right|_{x=a} = b^a \cdot \text{tangent slope at } (0,1)$$

Experimentally we find this tangent slope. $y = 2^x$

h	$\frac{2^h - 1}{h}$
0.01	0.6956...
0.001	0.6933...
0.0001	0.6931...
0.00001	0.6931...

tangent slope at $(0,1)$ of 2^x

is approximately $0.6931\ldots < 1$

$y = 4^x = 2^{2x}$

h	$\frac{4^h - 1}{h}$
0.01	1.3959...
0.001	1.3873...
0.0001	1.3864...
0.00001	1.3863......

tangent slope at $(0,1)$ is $1.3863\ldots$

$= 2\cdot(0.6931\ldots)$

graph of $y = 4^x = 2^{2x}$ is gotten graph of $y = 2^x$ by compressing by $\frac{1}{2}$ in horizontal direction

There is remarkable number e so that tangent slope of $y = e^x$ at $(0,1)$ is 1. So $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$, and derivative of e^x is $1e^x$

e is between 2 and 4

$\frac{de^x}{dx} = e^x$

$e = 2.718281828\ldots$

Once we know $(e^x)' = e^x$ we can find derivatives of other exponentials.

$$(b^x)' = (e^{(\log_e b)x})' \qquad b = e^{\log_e(b)}$$

The function $e^{(\log_e b)x}$ is composition of e^u outside function

$u = (\log_e b)x$ inside function

By chain rule

$$(b^x)' = \left(e^{(\log_e b)x}\right) \cdot (\log_e b) = (b^x) \cdot (\log_e b)$$

The factor $\log_e b$ is also tangent slope at $(0,1)$.

$(\log_e 2) = 0.6931 \qquad (2^x)' = 2^x \log_e(2)$

$(\log_e 4) = 1.3863 \qquad (4^x)' = 4^x \log_e(4)$

$(\log_e 10) = 2.302 \qquad (10^x)' = 10^x \log_e(10)$

WW4 #10 Find derivative of $y = \frac{e^{5x}}{x^7+1}$.

Use quotient rule, and fact $(e^x)' = e^x$.

$e^{5x} = e^u$

$u = 5x$

$$\frac{dy}{dx} = \frac{(5e^{5x})(x^7+1) - e^{5x}(7x^6+0)}{(x^7+1)^2}$$

$(e^{5x})' = e^u \cdot 5$

#11 Find derivative $w = (t^2+5t)(1-e^{-2t})$

$e^{-2t} = e^u$

$u = -2t$

Use product rule, and fact $(e^x)' = e^x$

$$\frac{dw}{dt} = (2t+5)(1-e^{-2t}) + (t^2+5t)(0 - e^{-2t}(-2))$$

Implicit functions Sometimes we are given relationship between two variables, but we do not have rule for one in terms of other

Examples ① $x^2+y^2=1$ circle

This relation is simple and we could solve for y in terms of x, or vice versa.

$y^2 = 1-x^2$, $y = +\sqrt{1-x^2}$ or $-\sqrt{1-x^2}$

② $x^3+y^2=6xy$. This is quadratic in y. Could use quadratic equation to get rule for y in terms of x

Get 2 functions.

(3) $x^3 + y^3 = 3xy^2 - x - 1$ Very difficult to solve for y in terms of x or vice versa

(4) $(x^2+y^2-1)^3 - x^2y^3 = 0$ is heart shaped 

see link at course web page.

Although we do <u>not</u> have formula for y in terms of x we can still find tangent slopes.

Implicit differentiation

WW5 #1. Curve $x^5 + 4xy + y^4 = 41$

contains point $(2,1)$.

Verify $(2,1)$ on curve $2^5 + 4\cdot 2\cdot 1 + 1^4 \overset{?}{=} 41$

$32 + 8 + 1 = 41$ ✓ so $(2,1)$ on curve.

Find tangent slope at $(2,1)$, and find tangent line.

Although we do not have formula for y in terms of x, we still think of it as function of x.

Take $x^5 + 4xy + y^4 = 41$, and take $\frac{d}{dx}$ of both sides

$$5x^4 + 4\cdot\left(1\cdot y + x\frac{dy}{dx}\right) + 4y^3\cdot\frac{dy}{dx} = \frac{d}{dx}(41) = 0.$$

Now solve for $\frac{dy}{dx}$: $\frac{dy}{dx}(4x + 4y^3) = -(5x^4 + 4y)$

$$\frac{dy}{dx} = \frac{-(5x^4 + 4y)}{(4x + 4y^3)}$$