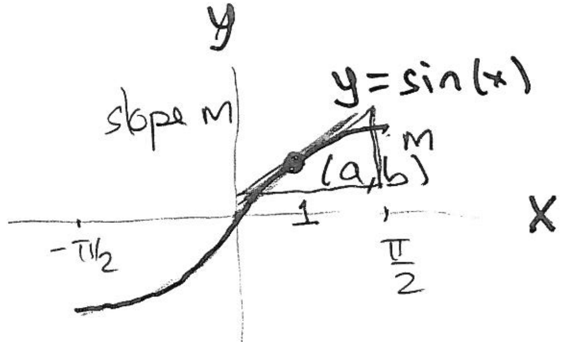
Derivative of inverse function

Example $y = f(x) = \sin x$ one-to-one on domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

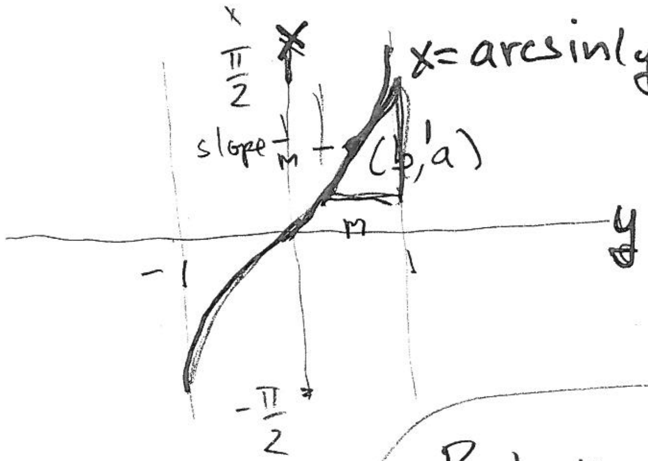
range of values is $-1 \le y \le 1$

Inverse function (called arcsin(y)) domain $-1 \le y \le 1$

range is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.



$\frac{dy}{dx} = \cos(x)$



$$\frac{d(\arcsin(y))}{dy} = \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$(\arcsin(y))' = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{\cos(x)}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

Since y is input to arcsin, want to express $\frac{1}{\cos(x)}$ in terms of input y.

But $y = \sin x$, so $\cos x = \sqrt{1-(\sin x)^2}$
$= \sqrt{1-y^2}$

So $\frac{d(\arcsin(y))}{dy} = \frac{1}{\sqrt{1-y^2}}$.

$y = \tan(x) = \frac{\sin(x)}{\cos(x)}$

So using rules for derivatives

$$\frac{dy}{dx} = \frac{\cos(x)\cos(x) - \sin(x)(-\sin x)}{(\cos x)^2}$$

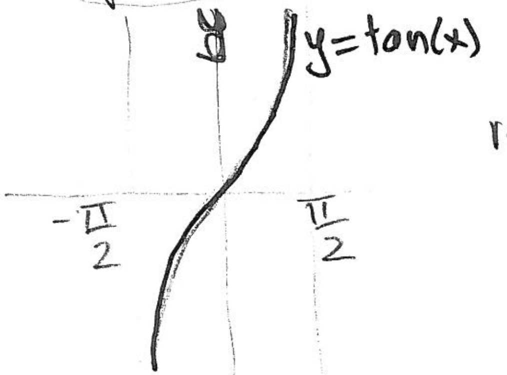
$$= \frac{1}{(\cos x)^2} \qquad \frac{1}{\cos x} = \sec x$$

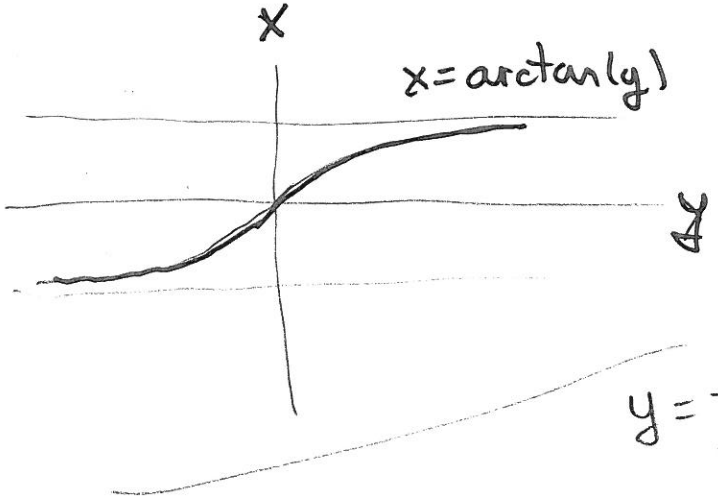
$$\frac{dy}{dx} = (\sec x)^2.$$

Find derivative of arctan

$y = \tan(x)$ domain (for one-to-one) $-\frac{\pi}{2} < x < \frac{\pi}{2}$

range $-\infty < y < \infty$





Derivative arctan $\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{\left(\frac{1}{(\cos x)^2}\right)} = (\cos x)^2$

Since input to arctan(y) is y, we want to put everything in terms of y.

$y = \frac{\sin(x)}{\cos(x)}$ so $y^2 = \frac{(\sin(x))^2}{(\cos(x))^2}$, so $y^2 + 1 = \frac{(\sin(x))^2}{(\cos(x))^2} + \frac{(\cos(x))^2}{(\cos(x))^2}$

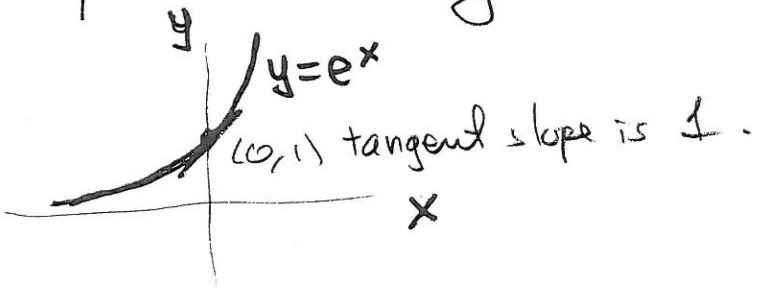
$= \left(\frac{1}{\cos x}\right)^2.$

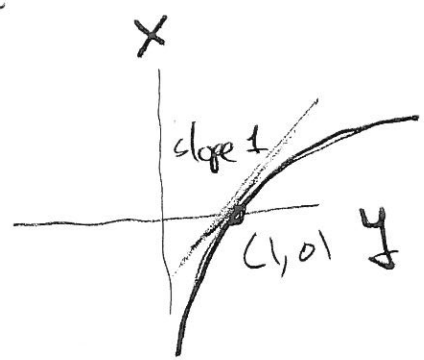
So $\frac{dx}{dx} = (\cos x)^2 = \frac{1}{y^2+1}$ | $(\arctan(y))' = \frac{1}{y^2+1}$

Exponential $y = f(x) = e^x$ $y = e^x$ $\frac{dy}{dx} = e^x$

$\log_e(x)$

Use notation $\ln(x)$





$$\frac{d(\ln(y))}{dy} = \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{e^x} = \text{write in terms of } y = \frac{1}{y}.$$

$$(\ln(y))' = \frac{1}{y}.$$

New basic derivatives

(1) $\frac{d(\arcsin(y))}{dy} = \frac{1}{\sqrt{1-y^2}}$ (3) $((\ln(y))' = \frac{1}{y}$.

(2) $\frac{d(\arctan(y))}{dy} = \frac{1}{y^2+1}$.

WW5 #2 Find derivative $\frac{dy}{dx}$ of

$$y = f(x) = \frac{x^3 (x-9)^7}{(x^2+5)^6}$$

powers, products, quotients
can find derivative
using rules for powers,
products, quotients

Trick logarithm ln converts products to sums
quotients to differences

Easier to take derivative of sum than of product and difference instead of quotient

Take ln( ) of both sides:

$$\ln(y) = \ln\left(\frac{x^3(x-9)^7}{(x^2+5)^6}\right) = \ln(x^3) + \ln(x-9)^7 - \ln(x^2+5)^6$$

$$\ln(y) = 3\ln(x) + 7\ln(x-9) - 6\ln(x^2+5)$$

Now take $\frac{d}{dx}$ of both sides. $\frac{d}{dx}(\ln(y)) = \frac{d}{dx}($ above expression $)$

$\frac{d(\ln(y))}{dx}$ is

$$\frac{1}{y} \cdot \frac{dy}{dx} = 3 \cdot \frac{1}{x} + 7 \cdot \frac{1}{(x-9)}(1-0) - 6\frac{1}{x^2+5} \cdot (2x+0)$$

$\frac{d(\ln(y))}{dy} = \frac{1}{y}$

So $\dfrac{dy}{dx} = y \cdot \left(\dfrac{1}{y} \dfrac{dy}{dx} \right) = y \cdot \left(3 \cdot \dfrac{1}{x} + \dfrac{7}{(x-9)} - \dfrac{6 \cdot 2x}{x^2+5} \right)$

$$\frac{dy}{dx} = \left(\frac{x^3 (x-7)^7}{(x^2+5)^6} \right) \cdot \left(\frac{3}{x} + \frac{7}{(x-9)} - \frac{12x}{x^2+5} \right)$$

This process is called logarithmic differentiation.

WW6 (opens next week) has problems with arctan, arcsin

Related rates Situation. Have 2 more quantities related by some equations, and quantities are function of a variable. There will be relations between the instantaneous rate of change of the quantities.

WW5 #8. Air being pumped into a ballon

At $r = 14\text{ cm}$, the instantaneous rate of change of the volume is $\frac{dV}{dt} = 60\ \text{cm}^3/\text{sec}$.

Find instantaneous rate of change of $\frac{dS}{dt}$ of surface area.

Quantities are radius r, surface area S, volume V.

All functions of time t.

Solution ① Relate $\frac{dV}{dt}$ to $\frac{dr}{dt}$. Use $V = \frac{4}{3}\pi r^3$

Take $\frac{d}{dt}$. We get $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt}$

Solve for $\left.\frac{dr}{dt}\right|_{r=14}$ using $r=14$, $\frac{dV}{dt} = 60$

$$\left.\frac{dr}{dt}\right|_{r=14} = \left.\left(\frac{dV}{dt}\right)\right|_{r=14} \frac{1}{4\pi r^2} = 60\ \frac{\text{cm}^3}{\text{sec}} \cdot \frac{1}{4\pi \cdot (14\text{ cm})^2}$$

2 Relate $\frac{dr}{dt}$ to $\frac{dS}{dt}$: $S = 4\pi r^2$.

Take $\frac{d}{dt}$ both sides to get $\frac{dS}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt}$

$$\left.\frac{dS}{dt}\right|_{r=14} = \cancel{4\pi} \cdot 2 \cdot (14\,cm) \cdot 60 \left(\frac{cm^3}{sec}\right) \cdot \frac{1}{\cancel{4\pi}(14\,cm)^2} \qquad \frac{cm^2}{sec}$$

$$= \frac{\cancel{2} \cdot 60}{\cancel{14}\,7} \frac{cm^2}{sec} = \frac{60}{7} \frac{cm^2}{sec}.$$