Derivative of inverse function

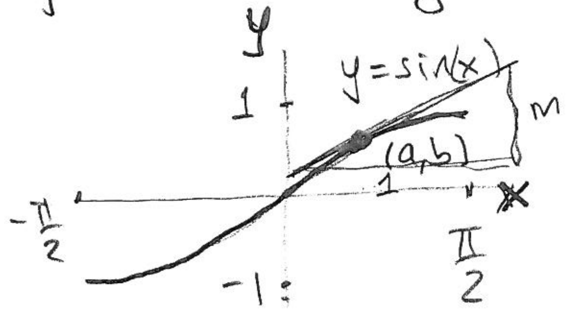
Example $y = f(x) = \sin x$ on domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

becomes one-to-one and its range is $-1 \le y \le 1$.

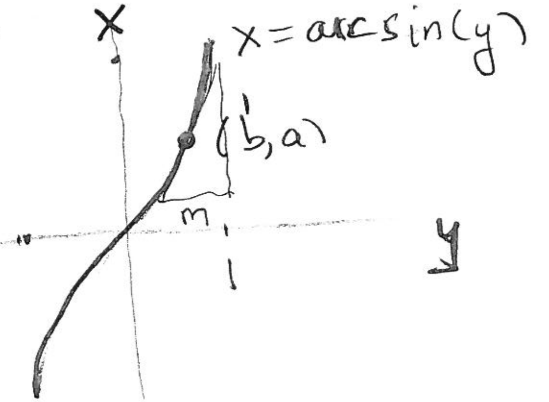
There is inverse function (arcsin) has domain $-1 \le y \le 1$

$x = \arcsin(y)$

Range of $\arcsin(y)$ is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$



Flip graph across 45° line



If the tangent slope at point (a,b) of sine graph is m

then the tangent slope at point (b,a) of arcsin graph is $\frac{1}{m}$

$y = \sin(x)$. $\frac{dy}{dx}$ = derivative of sin. $x = \arcsin(y)$ $\frac{dx}{dy}$ = derivative of arcsin

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$y = \sin x \rightsquigarrow \dfrac{dy}{dx} = \cos x$

So $x = \arcsin(y) \rightsquigarrow \dfrac{dx}{dy} = \dfrac{1}{\left(\frac{dy}{dx}\right)} = \dfrac{1}{\cos x}$

y variable, x is function of y.

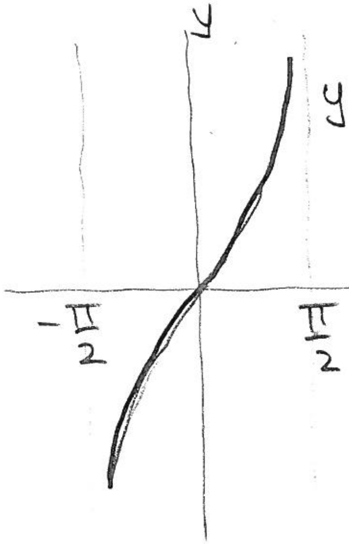
Like to write $\dfrac{1}{\cos x}$ in terms just of y.

Solve for $\cos x$ in terms of y $(=\sin x)$

so $\cos x = \sqrt{1-(\sin x)^2} = \sqrt{1-y^2}$

Therefore $\dfrac{dx}{dy} = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1-y^2}}$ $\quad (\arcsin(y))' = \dfrac{1}{\sqrt{1-y^2}}$.

Derivative of arctan



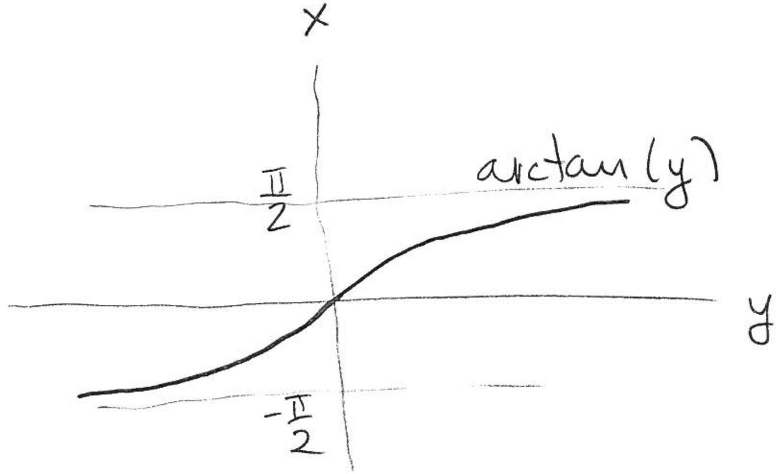
$y = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$

one-to-one

X

range $-\infty < y < \infty$



Derivative of $x = \arctan(y)$ is given as

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{\left(\frac{1}{\cos(x)}\right)^2} = (\cos x)^2.$$

Write $(\cos x)^2$ just in terms of $y = \dfrac{\sin x}{\cos x}$

$$y^2 = \left(\frac{\sin x}{\cos x}\right)^2 = \frac{(\sin x)^2}{(\cos x)^2}, \quad y^2+1 = \frac{(\sin x)^2}{(\cos x)^2} + \frac{(\cos x)^2}{(\cos x)^2}$$

$$= \frac{1}{(\cos x)^2}$$

so $(\cos x)^2 = \dfrac{1}{y^2+1}$. Therefore

$$\frac{dx}{dy} = \frac{1}{y^2+1} \qquad (\arctan y)' = \frac{1}{y^2+1}$$

$$y = \frac{\sin(x)}{\cos(x)}$$

$$\frac{dy}{dx} = \frac{(\sin(x))'\cos(x) - (\sin x)(\cos x)'}{(\cos x)^2}$$

$$= \frac{(\cos x)^2 - (\sin x)(-\sin x)}{(\cos x)^2}$$

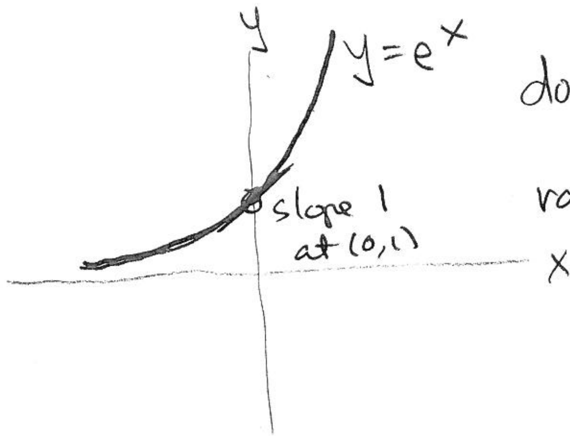
$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} =$$

$$= \frac{1}{(\cos x)^2} = (\sec x)^2.$$

Exponential function $y = e^x$ $\quad \frac{dy}{dx} = e^x$

$\log_e(\ ) = \ln(\ )$

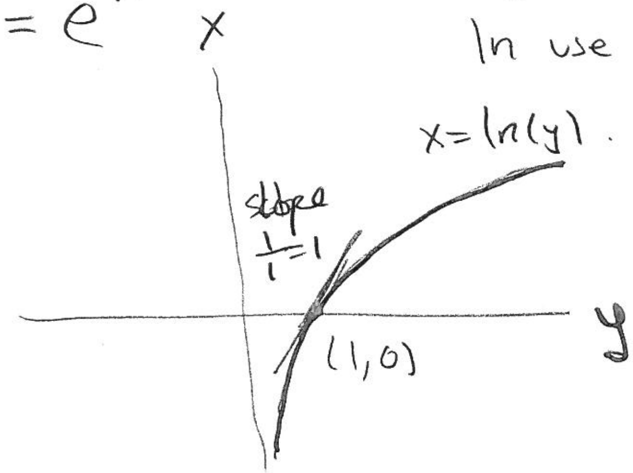
ln use to denote $\log_e$.



domain $-\infty < x < \infty$

one-to-one

range $0 < y < \infty$



Derivative of $x = \ln(y)$ is

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{e^x} = \frac{1}{y}. \qquad (\ln(y))' = \frac{1}{y}.$$

More basic derivatives

(a) $(\arcsin(y))' = \frac{1}{\sqrt{1-y^2}}$

(b) $(\arctan(y))' = \frac{1}{y^2+1}$

(c) $(\ln(y))' = \frac{1}{y}$.

WW5 #2 Find derivative of $\frac{dy}{dx}$.

$$y = f(x) = \frac{x^3(x-9)^7}{(x^2+5)^6}.$$ (quotient, could use quotient rule)

Trick. The ln function changes products to sums
quotients to differences

Take the ln of y. It changes products to sums,
quotients to differences

$$\ln(y) = \ln\left(\frac{x^3(x-9)^7}{(x^2+5)^6}\right) = \ln(x^3) + \ln(x-9)^7 - \ln(x^2+5)^6$$
$$= 3\ln(x) + 7\ln(x-9) - 6\ln(x^2+5).$$

Take $\frac{d}{dx}$ of both sides and use fact $(\ln(u))' = \frac{1}{u}$.

$$(\ln(y))' = \frac{1}{y}\cdot\frac{dy}{dx} \text{ by chain rule.}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{7}{(x-9)} + \frac{6(2x)}{x^2+5}$$

$$\frac{dy}{dx} = y\left(\frac{3}{x} + \frac{7}{(x-9)} + \frac{6\cdot(2x)}{x^2+5}\right)$$

$$\frac{d}{dx}(3\ln(x)) = \frac{3}{x}$$

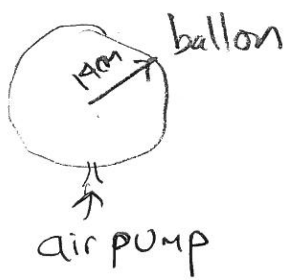
$$\frac{d}{dx}(7\ln(x-9)) = \frac{7}{(x-9)}\cdot 1$$

$$\frac{d}{dx}(6\ln(x^2+5)) = 6\frac{1}{x^2+5}\cdot(2x+0)$$

Related Rates Sometimes two or more quantities are related by an equation, and the quantities are functions of some other variable/parameter.

For related rates we want to relate the derivative with respect to the variable/parameter.

Example WW5 #8 Air pumped into spherical ballon and the instantaneous rate of change in volume when radius $r = 14\text{cm}$ is $\frac{dV}{dt} = 60\ \text{cm}^3/\text{sec}$. Determine the instantaneous rate of change of surface area $\frac{dS}{dt}$ at that instant.



$V = \frac{4}{3}\pi r^3$ @ $r = 14\text{ cm}$ $\frac{dV}{dt} = \frac{60\text{ cm}^3}{\text{sec}}$

$S = 4\pi r^2$

Find $\frac{dS}{dt}$

Quantities
V volume
S surface
each is function of t.

Solution 1st We use $V=\frac{4}{3}\pi r^3$. to relate $\frac{dV}{dt}$ to $\frac{dr}{dt}$ @ $r=14$cm.

2nd We use $S=4\pi r^2$ to relate $\frac{dS}{dt}$ to $\frac{dr}{dt}$.

① $V=\frac{4}{3}\pi r^3$. Take $\frac{d}{dt}$ to get $\frac{dV}{dt}=\frac{4}{\cancel{3}}\pi\cdot\cancel{3}r^2\cdot\frac{dr}{dt}$ $(r(t))^3$

So at $r=14$cm, $\frac{dV}{dt}=60\ \text{cm}^3/\text{sec}$ we get

$$\frac{dr}{dt}=\left(\frac{dV}{dt}\right)\frac{1}{4\pi(14)^2\text{cm}^2}=60\frac{\text{cm}^3}{\text{sec}}\cdot\frac{1}{4\pi(14\,\text{cm})^2}\quad\left(\frac{\text{cm}}{\text{sec}}\right)$$

② $S=4\pi r^2$, Take $\frac{d}{dt}$ to get $\frac{dS}{dt}=4\pi\cdot2\cdot r\cdot\frac{dr}{dt}$

$$\frac{dS}{dt}=\cancel{4\pi}\cdot2\cdot(14)\cdot60\frac{1}{\cancel{4\pi}(14)^2}\frac{\text{cm}^2}{\text{sec}}$$

$$=\left(\frac{120}{14}\right)\frac{\text{cm}^2}{\text{sec}}.$$