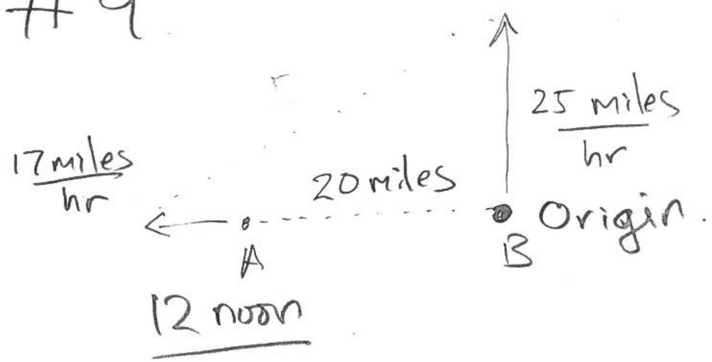
Week 7 Monday L03 2pm

Related rates Situation where we have several quantities $Q_1, Q_2, \ldots$ which are function of some parameter s and equation relating them. $\frac{d}{ds}$ allows us to relate the rate of changes $\frac{dQ_1}{ds}, \frac{dQ_2}{ds}$, etc.

WW5 #9

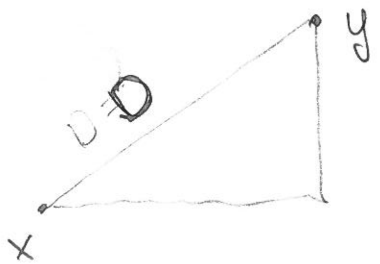


Determine (instantaneous) rate of change of distance between ships at 7pm

x = horizontal distance of ship A from origin $x(t) = 20 + 17t$

y = vertical distance of ship B from origin $y(t) = 0 + 25t$

Distance between two ships is $D = (x^2+y^2)^{1/2}$



Find $\frac{dD}{dt}$

x, y, D related by $D = (x^2+y^2)^{1/2}$

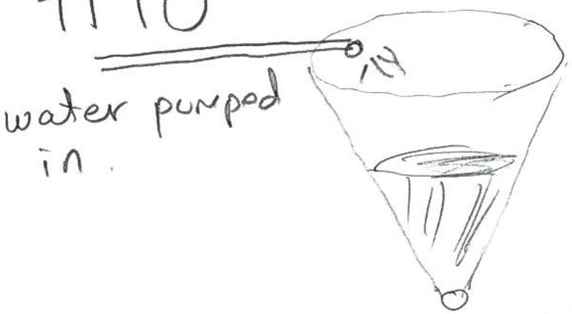
Take $\frac{d}{dt}$ to get

$$\frac{dD}{dt} = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}\left(\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}\right)$$

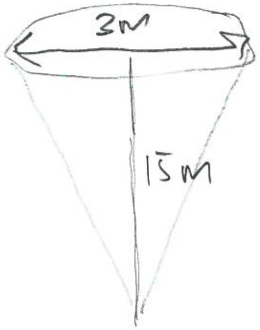
At 7pm: $x = 20 + 17 \cdot 7$, $y = 25 \cdot 7$, $\frac{dx}{dt} = 17$, $\frac{dy}{dt} = 25$

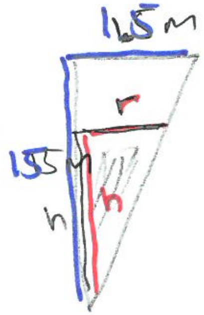
Plug in to get $\left.\frac{dD}{dt}\right|_{t=7pm} = \left((20+17\cdot 7)^2 + (25\cdot 7)^2\right)^{-1/2}\left((20+17\cdot 7)\cdot 17 + (25\cdot 7)\,25\right)$

#10



leak $\hookrightarrow$ $13{,}900 \text{ cm}^3/\text{min}$.





$h(t)$ = water height at time t

$r(t)$ = radius of water top.

By similar triangles

$$\frac{1.5\text{ m}}{15\text{ m}} = \frac{1}{10} = \frac{r}{h}$$

$$r = \frac{1}{10}h.$$

Given $\frac{dh}{dt} = 26 \text{ cm}/\text{min}$ when $h = 3$ meters $= 300$ cm.

Find rate water is being pumped in.

V = volume of water. $\frac{dV}{dt}$ = (instantaneous) rate of change of water

$$\frac{dV}{dt} = \left(\text{rate water pumped in}\right) - \underbrace{\left(\text{rate of leak}\right)}_{13{,}900\text{ cm}^3/\text{min}}$$

So rate water pumped in $= \frac{dV}{dt} + 13{,}900$ $\left(\frac{\text{cm}^3}{\text{min}}\right)$

So we need to find $\frac{dV}{dt}$. Volume of water is $\frac{1}{2}$ base $\cdot$ height $= \frac{1}{3}\pi r^2 \cdot h$

$$= \frac{1}{3}\pi\left(\frac{1}{10}h\right)^2 \cdot h$$

$$V = \frac{1}{3}\pi\frac{1}{100}h^3.$$

Since $V = \frac{1}{3}\pi\frac{1}{100}h^3$, we $\frac{d}{dt}$ both sides to get

$$\frac{dV}{dt} = \frac{1}{3}\pi\frac{1}{100}3h^2 \cdot \frac{dh}{dt}$$

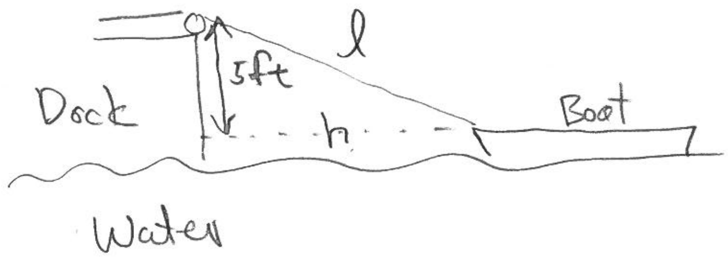
$h = 300$ cm, $\frac{dh}{dt} = 26 \text{ cm}/\text{min}$

We get

$$\text{rate water pumped in} = \frac{1}{\cancel{3}}\pi\left(\frac{1}{100}\right)\cancel{3}\cdot(300\text{ cm})^2\cdot\frac{26\text{ cm}}{\text{min}} + 13{,}900\ \frac{\text{cm}^3}{\text{min}}$$

$$= \left(\frac{\pi}{100}(300)^2\cdot 26 + 13{,}900\right)\text{cm}^3/\text{min}$$

#11



Relationship between l and h is $l^2 = 5^2 + h^2$

$\frac{dl}{dt} = -18\ ft/min$ (rate rope being pulled in).

Find $\frac{dh}{dt}$ rate of change of horizontal distance when $l = 90\ ft$.

$h = (l^2 - 5^2)^{1/2}$. Take $\frac{d}{dt}$ of both sides to get

$$\frac{dh}{dt} = \frac{1}{2}(l^2 - 5^2)^{-1/2} \cdot \left(2l\frac{dl}{dt} - 0\right)$$

When $l = 90$, and $\frac{dl}{dt} = -18\ ft/min$ we get

$$\left.\frac{dh}{dt}\right|_{l=90ft} = \frac{1}{\cancel{2}}(90^2 - 5^2)^{-1/2} \cdot \left(\cancel{2} \cdot 90\,(-18\ ft/min)\right)$$

$$= \frac{-90 \cdot 18}{(90^2 - 5^2)^{1/2}}\ ft/min.$$

Applications of logarithms and exponentials

WW6 #1 Bank interest. Bank pays annual interest of r%
paid quarterly (4 times a year).
Initial deposit of 3500 left to grow for 7 years.
At the end the balance is 4854.2. Find interest rate r.

A_0 = initial deposit. After one quarter the amount is

$$\underbrace{A_1 = A_0}_{\text{initial}} + \underbrace{\left(\frac{r}{4}\right) A_0}_{\text{interest}} = A_0\left(1+\frac{r}{4}\right)$$

After another quarter the amount

$$A_2 = A_1\left(1+\frac{r}{4}\right) = A_0\left(1+\left(\frac{r}{4}\right)\right)^2$$

After 7 years (28 quarters) the amount is $A_{28} = A_0\left(1+\left(\frac{r}{4}\right)\right)^{28}$

$$4854.2 = 3500\left(1+\frac{r}{4}\right)^{28}$$

$$\left(1+\frac{r}{4}\right)^{28} = \frac{4834.2}{3500}$$

We solve for r using logarithm.
Take natural logarithm ln of both sides

$$\ln\left(\left(1+\frac{r}{4}\right)^{28}\right) = \ln\left(\frac{4854.2}{3500}\right)$$

$$28\ln\left(\left(1+\frac{r}{4}\right)\right) = \ln\left(\frac{4854.2}{3500}\right)$$

$$\ln\left(1+\frac{r}{4}\right) = \frac{1}{28}\ln\left(\frac{4854.2}{3500}\right).$$

$$1+\frac{r}{4} = \exp\left(\ln\left(1+\frac{r}{4}\right)\right) = \exp\left(\frac{1}{28}\ln\left(\frac{4854.2}{3500}\right)\right).$$

$$r = 4\left(\exp\left(\frac{1}{28}\ln\left(\frac{4854.2}{3500}\right)\right)-1\right)$$

Then multiply by 100% to get percentage.