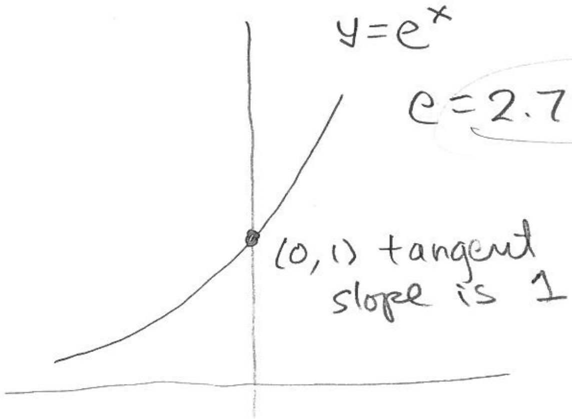
Continuous interest and the remarkable number e



Suppose bank offers <u>100%</u> annual interest and it compounded (yearly, semiannually, quarters, monthly, daily). Suppose initial deposit of 1.

	times per year	amount
	1	$1+1=2$
6month	2	$(1+\frac{1}{2})(1+\frac{1}{2})=2.25$
3month	4	$(1+\frac{1}{4})^4 = 2.44106\ldots$
monthly	12	$(1+\frac{1}{12})^{12} = 2.613035\ldots$
daily	365	$(1+\frac{1}{365})^{365} = 2.714567\ldots$
minute	$365\times 24 \times 60$	$(1+\frac{1}{365\times 24\times 60})^{365\times 24\times 60} = 2.71827\ldots$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \text{limit exists} = 2.71828\ldots \quad (\text{the number } e).$$

continuous interest

For interest rate r: $\lim\limits_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$.

Why? Take ln of $\left(1+\frac{r}{n}\right)^n$, so $y_n = \ln\left(1+\frac{r}{n}\right)^n = n\ln\left(1+\frac{r}{n}\right)$.

We show $\lim\limits_{n\to\infty} y_n = r$, then $\lim\limits_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$.

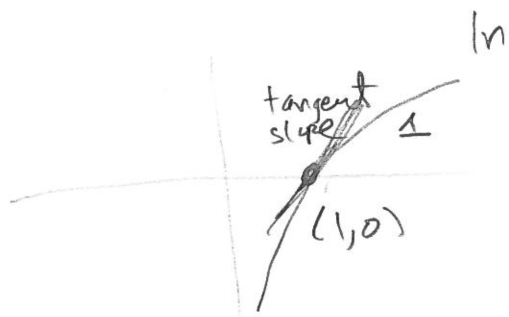
$$y_n = n\ln\left(1+\frac{r}{n}\right) = \frac{\ln\left(1+\frac{r}{n}\right) - \ln(1)}{\frac{1}{n}} = r\cdot\left(\frac{\ln\left(1+\frac{r}{n}\right)-\ln(1)}{\left(\frac{r}{n}\right)}\right).$$

As $n\to\infty$, the quantity $\frac{r}{n}\to 0$, so

$$\lim_{n\to\infty} y_n = \lim_{h\to 0} r\cdot\left(\frac{\ln(1+h)-\ln(1)}{h}\right) = r\cdot\lim_{h\to 0}\left(\frac{\ln(1+h)-\ln(1)}{h}\right)$$

$= r\cdot$ tangent slope at $(1,0)$

$= r\cdot 1$.



#2 Population growth If $P(t)$ = population of some species.

Model growth:

$$\frac{dP}{dt} = k \cdot P \qquad k \text{ a proportionality constant.}$$

$0 < k < 1$.

Solutions. $(e^t)' = e^t$

$(e^{kt})' = (e^{kt}) \cdot k$

The function e^{kt} is a solution of $\frac{dP}{dt} = kP$.

Also Ce^{kt} is a solution (2 unknowns k, C).

	t	$P(t)$
(1980)	0	25 million
(1990)	10	65 million

These two data points allow us to solve for unknowns k, C.

$25 = P(0) = Ce^{k\cdot 0} = C \cdot e^0 = C \quad C = 25$

$65 = P(10) = Ce^{k\cdot 10} = 25e^{k\cdot 10}$

solve for k: $e^{k\cdot 10} = \frac{65}{25}$, so $k\cdot 10 = \ln\left(\frac{65}{25}\right)$ so,

$k = \frac{1}{10}\ln\left(\frac{13}{5}\right)$

So $P(t) = 25 \exp\left(\left(\frac{1}{10}\ln\left(\frac{13}{5}\right)\right) t\right)$.

(b) Find population in year 2000 $(t = 20)$

$$P(20) = 25 \exp\left(\left(\frac{1}{10}\ln\left(\frac{13}{5}\right)\right) \cdot 20\right)$$

(c) Find doubling time. Exponentials change addition to multiplication. There is T_0 so that

$$P(t + T_0) = 2P(t)$$ — a shift in time by T_0, doubles population

$$Ce^{K(t+T_0)} = 2Ce^{Kt}$$

so $e^{KT_0} = 2$. Solve for T_0: $KT_0 = \ln(2)$

$$T_0 = \frac{\ln(2)}{K} = \frac{\ln(2)}{\left(\frac{1}{10}\ln\left(\frac{13}{5}\right)\right)}$$

#4 Bacteria growth $P(0) = 400$

$P(8) = 8000$

$\frac{dP}{dt} = KP$ solutions $C \cdot e^{Kt}$ with C, K unknown.

(a) Find $P(t) = Ce^{Kt}$ (find c, K) based on above data

$400 = P(0) = C \cdot e^{K \cdot 0} = C$ $\quad \underline{C = 400}$

$8000 = P(8) = 400\, e^{K \cdot 8}$ solve for K

$\frac{8000}{400} = e^{K \cdot 8}$ so $K \cdot 8 = \ln(20)$ $\quad 20 = \frac{80}{4}$.

$K = \frac{1}{8} \ln(20)$

$P(t) = 400 \exp\left(\frac{1}{8} \ln(20)\, t \right)$

(b) Find $P(9)$. Plug in $t = 9$.

$K = \frac{1}{8} \ln(20)$

(c) Find $P'(9)$. Use $\frac{dP}{dt} = K \cdot P$. $\quad P = P(9) = 400 \exp\left(\frac{1}{8} \ln(20)\, 9 \right)$

$= \left(\frac{1}{8} \ln(20) \right) \cdot 400 \exp\left(\frac{1}{8} \ln(20)\, 9 \right).$

(d) Determine when $P = 30{,}000$

Solve $P(t) = 30000$ for t

$$30000 = 400 \exp\left(\left(\tfrac{1}{8} \ln(20)\right) t \right)$$

$$75 = \frac{300\cancel{00}}{4\cancel{00}} = \exp\left(\tfrac{1}{8} \ln(20)\, t \right)$$

Take ln both sides

$$\ln(75) = \tfrac{1}{8} \ln(20)\, t$$

$$t = \frac{\ln(75)}{\left(\tfrac{1}{8}\right) \ln(20)} = \frac{8 \ln(75)}{\ln(20)}.$$

#3. Country has gasoline reserves of $167 \cdot 10^6$ gallons.
Current yearly consumption is 10^6 gallons.
Consumption increasing 4% year.

$$10^6 \quad 10^6(1.04) \quad 10^6(1.04)^2 \quad \cdots \quad 10^6(1.04)^{n-1}$$

1st 2nd 3rd ⋯ n

(a) How long do reserves last? We set

$$\cancel{10^6} + \cancel{10^6}(1.04) + \cancel{10^6}(1.04)^2 + \cdots + \cancel{10^6}(1.04)^{n-1} = 167 \cdot \cancel{10^6}$$

Solve for $\underline{n}$. $1 + (1.04) + (1.04)^2 + \cdots + (1.04)^{n-1} = 167.$

$$\frac{1.04^n - 1}{1.04 - 1} = 167, \text{ so } 1.04^n - 1 = 167 \times (0.04)$$

$$1.04^n = 167 \times (0.04) + 1$$

Take ln: $\ln(1.04)^n = \ln(167 \times 0.04 + 1)$

$$n \ln(1.04) = \ln(167 \times 0.04 + 1)$$

$$n = \frac{\ln(167 \times 0.04 + 1)}{\ln(1.04)}$$

$= 51.978$ years.
(much less than 167)

(b) Assuming reserves were doubled to $2 \times 167 \times 10^6$ gallons.

How many years does this last?

In previous answer, we change 167 to $2 * 167$. So

double the reserves lasts

$$n = \frac{\ln(2 * 167 \times 0.04 + 1)}{\ln(1.04)} = 67.9346 \text{ years.}$$

(again much less than 167).