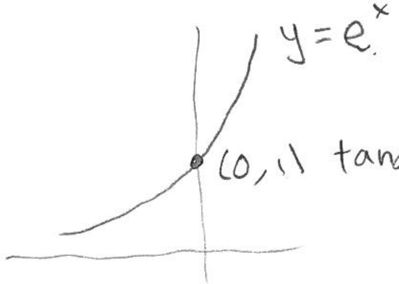
Continuous interest and the (remarkable) number e



Suppose a bank gives 100% annual interest. Suppose initial deposit is 1

# payments	amount at end of 1 year
1	$2 = 1+1$
(6 months) 2	$1(1+\frac{1}{2})(1+\frac{1}{2}) = (1.5)^2 = 2.25$
(3 months) 4	$(1+\frac{1}{4})^4 = 2.4414\ldots$
(monthly) 12	$(1+\frac{1}{12})^{12} = 2.61303\ldots$
(daily) 365	$(1+\frac{1}{365})^{365} = 2.71456\ldots$
(hourly) 365·24	$(1+\frac{1}{365\cdot 24})^{365\cdot 24} = 2.718126\ldots$
(by minute) 365·24·60	$(1+\frac{1}{365\cdot 24\cdot 60})^{365\cdot 24\cdot 60} = 2.71827\ldots$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \stackrel{?}{=}$$

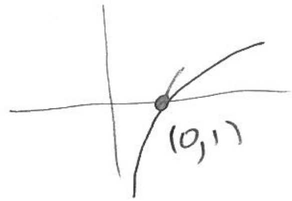
answer is the remarkable number $e = 2.71828\ldots$

Letting $n\to\infty$ is called continuous interest

Suppose the annual rate of interest is r.

If compounded n times, the multiplication factor is

$$\left(1+\frac{r}{n}\right)^n$$



The limit $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n$ equals e^r.

Why. Let $y_n = \ln\left(1+\frac{r}{n}\right)^n$. If we show $\lim_{n\to\infty} y_n = r$, then

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r.$$

But $y_n = n\ln\left(1+\frac{r}{n}\right) = \dfrac{\ln\left(1+\frac{r}{n}\right)-\ln(1)}{\frac{1}{n}} = r\cdot\left(\dfrac{\ln\left(1+\frac{r}{n}\right)-\ln(1)}{\frac{r}{n}}\right)$

As $n\to\infty$, the quantity $h=\frac{r}{n}\to 0$, so

$$\lim_{n\to\infty} y_n = \lim_{h\to 0} r\cdot\left(\frac{\ln(1+h)-\ln(1)}{h}\right) = r\cdot \text{tangent slope of } \ln \text{ at } (1,0)$$

$$= r\cdot 1.$$

WW6 #2 population is growing. $P(t)$ = population at time t.

We model this as $\frac{dP}{dt} = \left(\begin{matrix}\text{proportionality}\\ \text{constant}\end{matrix}\right) \cdot P$. $\frac{1}{20}P$.

Suppose

t	$P(t)$
$t=0$	25 million
$t=10$	65 millions

Solve the equation for growth.

$$\frac{dP}{dt} = K \cdot P.$$

As solution

$$P(t) = Ce^{Kt}$$

Two unknowns C, K.

$(e^t)' = e^t$

$(e^{Kt})' = e^{Kt} \cdot K$

From data points

$$25 = P(0) = Ce^{K\cdot 0} = C$$

$$65 = P(10) = C \cdot e^{K\cdot 10} = 25 \cdot e^{K\cdot 10}$$

$\frac{65}{25} = e^{K\cdot 10}$ take ln of both sides $K \cdot 10 = \ln\left(\frac{65}{25}\right)$

$$K = \frac{1}{10}\ln\left(\frac{65}{25}\right).$$

So $p(t) = 25 \cdot \exp\left(\frac{1}{10}\ln\left(\frac{13}{5}\right)t\right)$.

When $t=20$, we have $p(20) = 25 \cdot \exp\left(\frac{1}{10}\ln\left(\frac{13}{5}\right)\cdot 20\right)$

$$= 25 \cdot \exp\left(2\ln\left(\frac{13}{5}\right)\right).$$

$$P(t) = C \cdot e^{kt}$$

$$P(t+T_0) = C \cdot e^{k(t+T_0)} = e^{kT_0} \cdot P(t)$$

each shift of time by T_0 results in multiplication by the factor e^{kT_0}.

Doubling time is the T_0 so that $e^{kT_0} = 2$.

Solve

$$2 = e^{\frac{1}{10}\ln\left(\frac{65}{25}\right) \cdot T_0}.$$

$$\ln(2) = \frac{1}{10}\ln\left(\frac{65}{25}\right) \cdot T_0$$

so

$$T_0 = \frac{\ln(2) \cdot 10}{\ln(65/25)}$$

#4 Bacteria growth

$P(t) =$ population

$P(0) = 400$

$P(8) = 8000$.

Model. $\frac{dP}{dt} = kP$. Has solutions Ce^{kt} where C, k are constants.

$400 = P(0) = C \cdot e^{k \cdot 0} = C$

$8000 = P(8) = 400 \cdot e^{k \cdot 8}$ solve for k

$20 = \frac{8000}{400} = e^{k \cdot 8}$ so $\ln(20) = k \cdot 8$

$k = \frac{1}{8} \ln(20)$.

so $P(t) = 400 e^{\frac{1}{8}\ln(20) \cdot t}$

(b) when $t = 9$, $P(9) = 400 e^{\frac{9}{8}\ln(20)}$.

(c) growth rate at $t = 9$. $\frac{dP}{dt} = k \cdot P = \left(\frac{1}{8}\ln(20)\right) 400 e^{\frac{9}{8}\ln(20)}$.

(d) Determine what time $P = 30{,}000$

Solve $P(t) = 30000 = 400 \exp\left(\frac{1}{8}\ln(20)\, t\right)$ for t

$$75 = \frac{300\cancel{00}}{4\cancel{00}} = \exp\left(\frac{1}{8}\ln(20)\, t\right)$$

$$\ln(75) = \frac{1}{8}\ln(20)\, t$$

so $$t = \frac{8 \cdot \ln(75)}{\ln(20)}$$

#3 Country has gasoline reserves of $167 \cdot 10^6$ gallons
Current consumption is 10^6 gallons/year, but it is growing at 4% a year.

(a) How long will reserves last?

$$\underset{\text{1st}}{10^6}, \quad \underset{\text{2nd}}{10^6(1.04)}, \quad \underset{\text{3rd}}{10^6(1.04)^2}, \ldots, \quad \underset{\text{nth.}}{10^6(1.04)^{n-1}},$$

Solve for n.

$$10^6 + 10^6(1.04) + 10^6(1.04)^2 + \cdots + 10^6(1.04)^{n-1} = 167 \cdot 10^6$$

$$1 + (1.04) + (1.04)^2 + \cdots + (1.04)^{n-1} = 167.$$

equals $\dfrac{1.04^n - 1}{1.04 - 1} = 167$, so $\dfrac{1.04^n - 1}{.04} = 167$

so $1.04^n - 1 = 167 \cdot (0.04)$, so $1.04^n = 167 \cdot (0.04) + 1$

$$n \ln(1.04) = \ln(167 \cdot (0.04) + 1)$$

$$n = \frac{\ln(167 \cdot 0.04 + 1)}{\ln(1.04)} = 51.978 \text{ years}.$$

(b) If reserves is $2 \cdot 167 \cdot 10^6$ gallons instead.

How long would it last?

In formula change $167 \rightsquigarrow 2 \cdot 167$.

$$n = \frac{\ln(2 \cdot 167 \cdot 0.04 + 1)}{\ln(1.04)} = 67.9346 \text{ years.}$$