Tangent lines and differentials

Suppose f is a "nice" function, and suppose we want to approximate the function near input a.

(i) approximation by constant $y = b$

Best constant approximation (near input a)

$y = f(a)$ value of function at a.

(ii) approximation by line $y = mx + b$.

Best line approximation is tangent line

$$y = f(a) + f'(a)(x-a)$$

(iii) approximation by parabola $y = ax^2 + bx + c$

Best parabola approximation is parabola

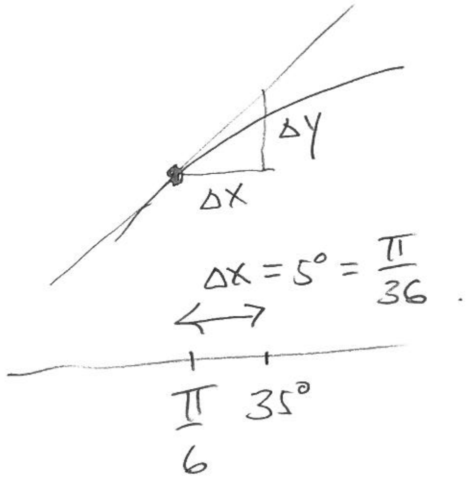
$$y = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Example Approximate $\sin(35°)$ using $\sin(30°) = \frac{1}{2}$

$$(\sin x)' = \cos x \qquad \cos(30°) = \frac{\sqrt{3}}{2}.$$

Use tangent line at input $30° = \left(\frac{\pi}{6}\right)$.

$$\left.\frac{dy}{dx}\right|_{30°} = \left.\cos x\right|_{30°} = \frac{\sqrt{3}}{2}.$$



$$35° = 30° + 5° = \left(\frac{\pi}{6}\right) + \left(\frac{\pi}{36}\right)$$

$$\frac{\Delta y}{\Delta x} \doteq \left.\frac{dy}{dx}\right|_{30°} = \frac{\sqrt{3}}{2}$$

$$\Delta y = \frac{\sqrt{3}}{2}\Delta x = \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right)$$

$$\sin(35°) = \sin(30°) + \Delta y = \frac{1}{2} + \left(\frac{\pi}{36}\right)\frac{\sqrt{3}}{2} = 0.57557.$$

Actual $\sin(35°) = 0.57358$

WW6 #5 Find approximation for $(27.4)^{1/3}$.

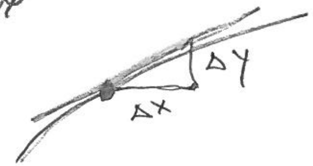
Take $y = f(x) = x^{1/3}$, $a = 27$, so $f(27) = 3$.

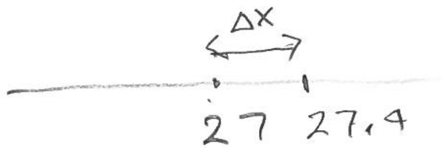
$$\frac{dy}{dx} = \frac{1}{3} x^{-2/3}$$

$\Delta x = 0.4$

$$\left.\frac{dy}{dx}\right|_{x=27} = \frac{1}{3}\left((27)^{-1/3}\right)^2 = \frac{1}{3}\frac{1}{3^2}$$

$$= \frac{1}{27}.$$ (tangent slope at $(27, 3)$)





$$\frac{\Delta y}{\Delta x} \doteq \left.\frac{dy}{dx}\right|_{x=27} = \frac{1}{27}$$

$$\Delta y \doteq \frac{1}{27}\Delta x = \frac{1}{27}(0.4)$$

Estimate for $(27.4)^{1/3}$ is $(27)^{1/3} + (0.4)\frac{1}{27} = 3.01481$

Actual value $(27.4)^{1/3} = 3.01474$

Differentials and error estimates

Example. Suppose we measure of radius of sphere to be 84cm with accuracy of ± 0.5cm.

(i) Compute surface area with estimate of error

$r = 84 \pm 0.5$ (cm) $\qquad \Delta r = \pm 0.5$

$S = 4\pi r^2 \qquad$ For $r = 84$cm, we get $S = 4\pi\, 84\ \text{cm}^2$

$= 88{,}668.2\ \text{cm}^2$

$$\frac{dS}{dr} = 4\pi \cdot 2r. \qquad \frac{\Delta S}{\Delta r} \doteq \frac{dS}{dr} = 4\pi(2\cdot 84\,\text{cm})$$

$$\Delta S \doteq 4\pi(2\cdot 84)\text{cm}\,(\pm 0.5\,\text{cm})$$

$$= 1{,}055\ \text{cm}^2$$

(ii) Compute volume with error

$$V = \frac{4}{3}\pi r^3 \qquad V = \frac{4}{3}\pi\,(84\text{ cm})^3 = 2,482,712\text{ cm}^3$$

$$\frac{dV}{dr} = \frac{4}{3}\pi\, 3r^2 = 4\pi r^2$$

$$\frac{\Delta V}{\Delta r} \doteq \frac{dV}{dr} = 4\pi r^2. \qquad \Delta V = 4\pi\,(84\text{cm})^2 \cdot \Delta r$$

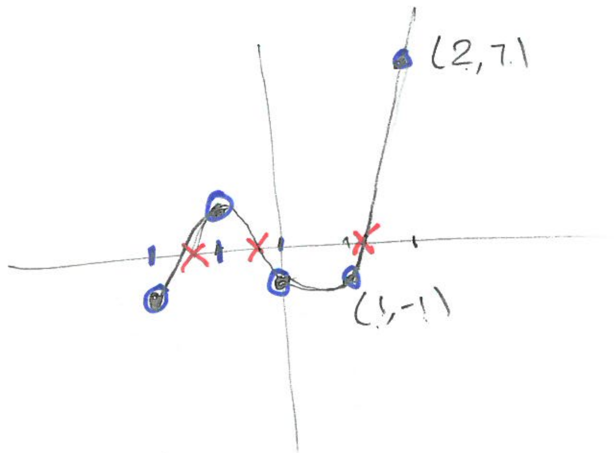
$$= 4\pi\,(84\text{cm})^2 \pm 0.5\text{ cm}$$

$$= \pm 4,433\text{ cm}^3$$

Newton's Method

Example $f(x) = x^3 + x^2 - 2x - 1$ Has 3 roots.

The function f is continuous

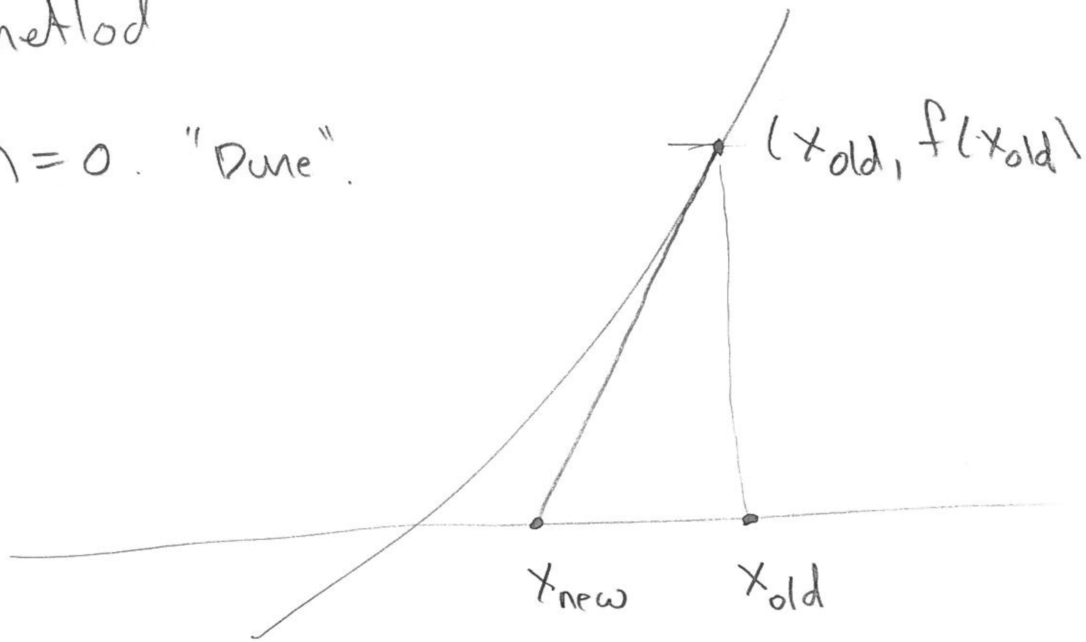


	x	f(x)	
	-2	-1	$-8+4+4-1$
root (	-1	1	$-1+1+2-1$
root (	0	-1	
	1	-1	$1+1-2-1$
root (	2	7	$8+4-4-1$

Intermediate Value Theorem If f is a continuous function on on interval $[a,b]$, and $f(a)$, $f(b)$ have opposite signs, then there is a point $\alpha \in [a,b]$ so that $f(\alpha) = 0$.

Newton's method

If $f(x_{old}) = 0$. "Done".



$x_{old} - x_{new}$ = horizontal change

$f(x_{old})$ = vertical change.

$$\frac{f(x_{old})}{x_{old} - x_{new}} = f'(x_{old})$$

Iteration

Solve for x_{new}: $\dfrac{f(x_{old})}{f'(x_{old})} = x_{old} - x_{new}$ so $x_{new} = x_{old} - \dfrac{f(x_{old})}{f'(x_{old})}$

x_0 = initial guess.

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_3)}$, ...

for $f(x) = x^3 + x^2 - 2x - 1$

$$I(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3+x^2-2x-1}{3x^2+2x-2} = \frac{2x^3+x^2+1}{3x^2+2x-2}$$

$x_0 = 2$, $x_1 = I(2) = 1.5$

$x_2 = I(x_1) = 1.2903$

$x_3 = I(x_2) = 1.2486$

$x_4 = I(x_3) = 1.24698$

$x_5 = I(x_4) = 1.24697$. accurate to 5 decimal places.