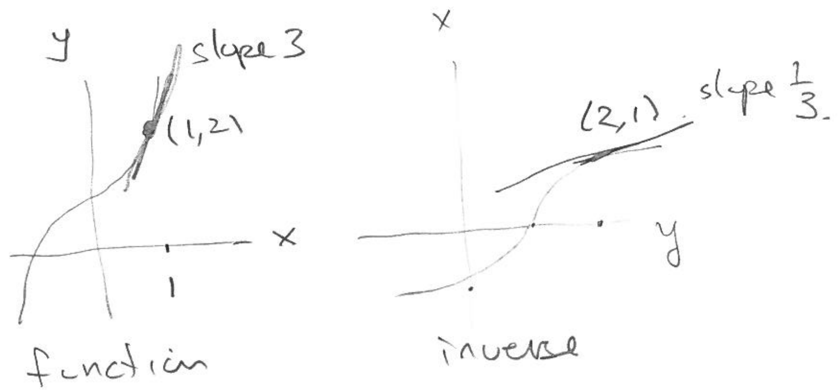
Week 8 Friday L03 2pm

#9 $y = f(x) = x^3 + 1$ is one-to-one. h inverse function

Find $h'(2)$. What is $h(2)$? $h(2)$ is the value x_2 so that $2 = f(x_2)$

$$2 = x_2^3 + 1 \text{ so } 1 = x_2^3 \text{ so } x_2 = 1$$

$$f(1) = 2 \text{ means } h(2) = 1.$$



$$h'(2) = \frac{1}{f'(1)} = \frac{1}{3}$$

$f'(x) = 3x^2 + 0$, so $f'(1) = 3$

Answer (a).

\#10 Velocity $v(t)$ given by graph.

When is runner slowing down most rapidly, so when is $\underline{\underline{v'(t)}}$ = instantaneous rate of change of v the lowest.

$t=5$ $v'(5)<0$ $t=10$ $v'(10)>0$

$t=15$ $v'(15)$ quite negative, $t=20$ $v'(20)=0$

$t=25$ $v'(25)<0$

Answer (c).

#12 (a) Express $f'(x)$ as limit.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}.$$

(b) For $f(x) = \frac{1}{\sqrt{x+1}}$ express $f'(a)$ as limit.

$$f'(a) = \lim_{x\to a} \left(\frac{\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{a+1}}}{x-a} \right)$$

To find limit

$$\frac{\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{a+1}}}{(x-a)} = \frac{\frac{\sqrt{a+1} - \sqrt{x+1}}{\sqrt{x+1}\sqrt{a+1}}}{(x-a)} \cdot \frac{\sqrt{a+1} + \sqrt{x+1}}{\sqrt{a+1} + \sqrt{x+1}} = \frac{\frac{\overbrace{(a+1) - (x+1)}^{\text{equals } a-x}}{\sqrt{x+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{x+1})}}{(x-a)}$$

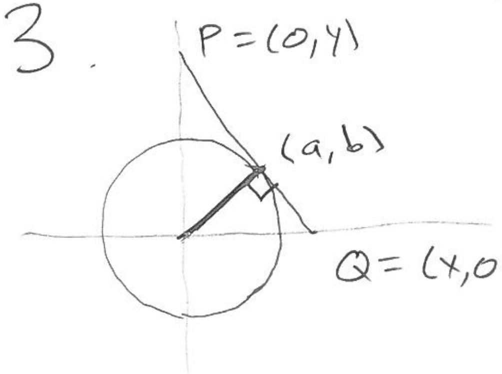
$$= \frac{-1}{\sqrt{x+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{x+1})} \xrightarrow[\text{as } x\to a]{} \frac{-1}{\sqrt{a+1}\sqrt{a+1}(\sqrt{a+1} + \sqrt{a+1})} = -\frac{1}{2}(a+1)^{-\frac{3}{2}}$$

$$f'(a) = -\frac{1}{2}(a+1)^{-3/2}$$

Check $f(x) = (x+1)^{-1/2}$ so $f'(x) = -\frac{1}{2}(x+1)^{-3/2}$ (HS) $= -\frac{1}{2}(x+1)^{-3/2}$ ✓

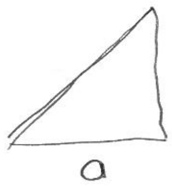
$x^2+y^2=2^2$

#13.



(a) Find equation relating $P=(0,y)$ and $Q=(x,0)$

Solution Write x, y in terms of a, b.



slope $\frac{b}{a}$ So slope of $\perp$ line $\overrightarrow{PQ}$ is $-\frac{1}{\left(\frac{b}{a}\right)} = -\frac{a}{b}$.

The P,Q line has slope $-\frac{a}{b}$ and goes through point (a,b).

$$y = b - \frac{a}{b}(x-a) = b - \frac{a}{b}(x-a)$$

So $P=(0,y)$ is when $x=0$, so $y = b - \frac{a}{b}(0-a) = b - \frac{a}{b}(-a) = b + \frac{a^2}{b}$

$$y = \frac{b^2+a^2}{b} = \frac{4}{b}$$ since (a,b) on circle $x^2+y^2=2^2$.

$Q=(x,0)$ is when $y=0$, so

$0 = b - \frac{a}{b}(x-a)$ $\quad 0 = b + \frac{a^2}{b} - \frac{a}{b}x$

$$\frac{a}{b}x = b + \frac{a^2}{b} = \frac{b^2+a^2}{b}, \quad x = \frac{4}{a}$$

$P=\left(0, \frac{4}{b}\right)$, $Q=\left(\frac{4}{a}, 0\right)$

(b) Find $\frac{dy}{dx}$ when $x=5$.

$a^2+b^2=4$ $\quad x=\frac{4}{a}, \quad y=\frac{4}{b}$.

$a=\frac{4}{x} \quad b=\frac{4}{y}$

$\left(\frac{4}{x}\right)^2+\left(\frac{4}{y}\right)^2=4$

$\frac{4}{x^2}+\frac{4}{y^2}=1$

$4x^{-2}+4y^{-2}=1$

Take $\frac{d}{dx}$ of $4x^{-2}+4y^{-2}=1$

$$\frac{d}{dx}\left(4x^{-2}+4y^{-2}\right) = \frac{d}{dx}(1) = 0$$

$$4(-2)x^{-3}+4(-2)y^{-3}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4(-2)x^{-3}}{4(-2)y^{-3}} = -\frac{y^3}{x^3} = 2.$$

$x=5$ $\quad \frac{4}{25}+\frac{4}{y^2}=1$

$$\frac{4}{y^2}=\frac{25-4}{25}=\frac{21}{25}$$

$$\frac{y^2}{4}=\frac{25}{21} \quad \boxed{y=2\frac{5}{\sqrt{21}}}$$

$$\left.\frac{dy}{dx}\right|_{x=5,\ y=2\frac{5}{\sqrt{21}}} = -\frac{\left(2\cdot\frac{5}{\sqrt{21}}\right)^3}{5^3}$$

$$= -\frac{8}{(21)^{3/2}} \quad \checkmark$$

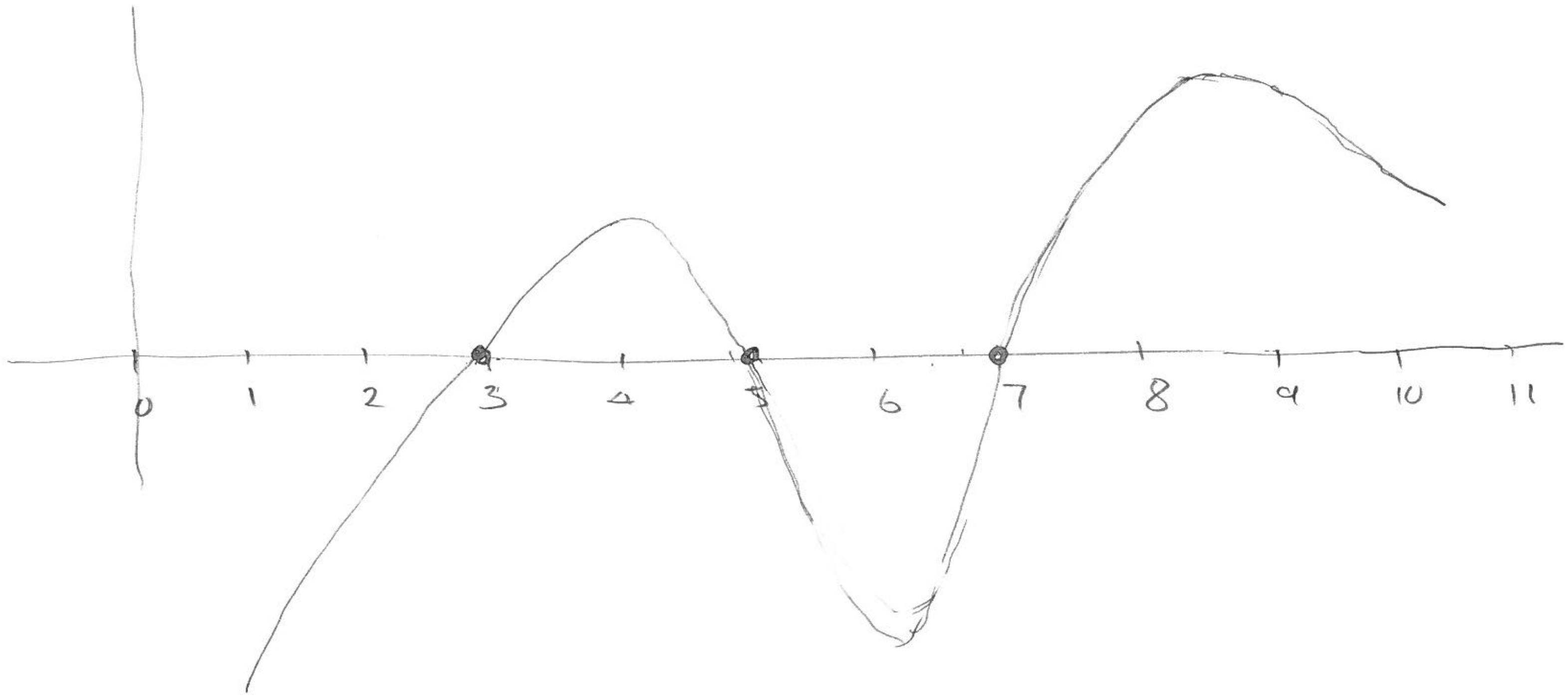
#14 Graph of function f show. Draw rough graph of derivative f'

Remember f'(a) is tangent slope at input a. So

where tangent slope is 0 $\Longrightarrow$ f'(a) is zero.

" " " " >0 $\Longrightarrow$ f'(a) > 0

" " " " <0 $\Longrightarrow$ f'(a) < 0



#15 For $f(x) = \sqrt{|x|}\,\tan(|x|^{3/2})$ determine if $f'(0)$ exists. Justify answer. Use definition of derivative.

$$f(0) = \sqrt{|0|}\,\tan(|0|^{3/2}) = 0 \cdot 0 = 0.$$

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt{|x|}\,\tan(|x|^{3/2})}{x}$$

$$\lim_{x \to 0^+} \frac{\sqrt{x}\,\tan(x^{3/2})}{x} = \lim_{x \to 0^+} \left(\frac{1}{x^{1/2}} \frac{\sin(x^{3/2})}{\cos(x^{3/2})} \right)$$

$$= \lim_{x \to 0^+} \frac{x}{\cos(x^{3/2})} \cdot \frac{\sin(x^{3/2})}{x^{3/2}} \longrightarrow \frac{0}{1} \cdot 1$$

$$= \frac{0}{1} \cdot 1 = 0$$

What about $\lim_{x \to 0^-}$? Easy to say function $f(x) = \sqrt{|x|}\,\tan|x|^{3/2}$ is EVEN, function x is ODD.

So $\dfrac{\sqrt{|x|}\,\tan(|x|^{3/2})}{x}$ is $\dfrac{\text{EVEN}}{\text{ODD}}$ is ODD. So $\lim_{x \to 0^-} \dfrac{f(x)}{x} = \lim_{x \to 0^+} \dfrac{-f(x)}{x} = -\lim_{x \to 0^+} \dfrac{f(x)}{x}$

$= -0$

$= 0.$