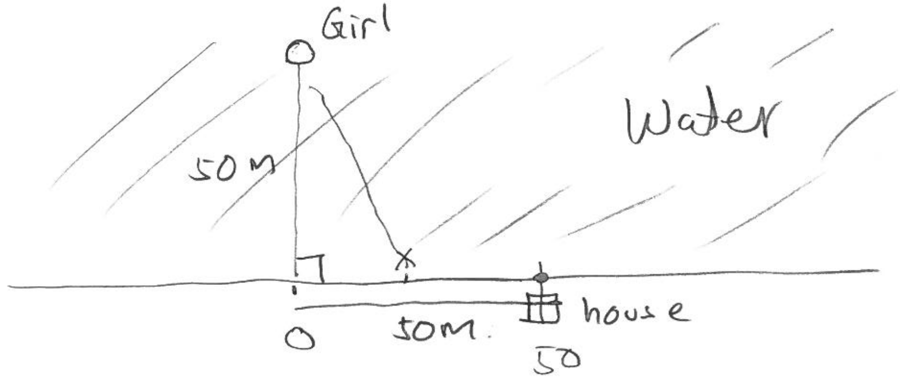
Maximum and minimum of functions

Example



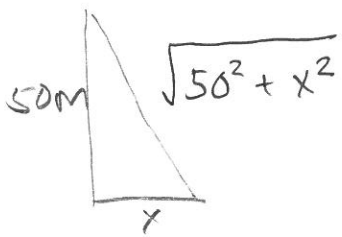
Girl wishes to swim/walk to house.

Can swim at 2 meters/sec

Can run/walk at 4 meters/sec

Where (x) should girl swim to and then walk so the TOTAL time is minimized?

Swim time



$$\frac{\sqrt{50^2+x^2}}{2} = \text{swim time}$$

Walk time

$$\frac{50-x}{4} = \text{walk/run time}$$

continuous

Total time is function $f(x) = \dfrac{\sqrt{50^2+x^2}}{2} + \dfrac{50-x}{4}$. Domain $\underbrace{0 \le x \le 50}_{\text{closed}}$

$f(0) = \frac{50}{2} + \frac{50}{4} = 50\left(\frac{3}{4}\right)$, $\quad f(50) = \frac{50\sqrt{2}}{2} + 0$

Properties of the total time function

$$f(x) = \frac{\sqrt{50^2+x^2}}{2} + \frac{50-x}{4}. \qquad \text{Domain } (0 \le x \le 50)$$

f is continuous and differentiable.

Definitions If f is a function on an interval $[a,b]$, or $(a,b]$, or $[a,b)$, or (a,b).

has an absolute (or global) minimum at input x_0 if the value $f(x_0)$ at x_0 is less than or equal all other values.

Similarly for maximum.

In above swim/walk function we seek absolute/global minimum.

A local minimum is a input x_0 so that the value $f(x_0)$ is less than or equal to value of nearby points

Extreme Value Theorem If f is a continuous function on a closed interval [a,b], then f has a absolute/global maximum and a absolute/global minimum.

This insures there are abs max, min.
Where do we look for them?

Local Extreme Value Theorem If f has local max or min an interior point c of the interval, and f is differentiable there, then $f'(c) = 0$.

To find local max min we look at 3 possible places.

1. interior points where $f' = 0$
2. interior points where f' does NOT exist
3. endpoints.

For the swim/walk function. To find local min we look at

① $f'(c) = 0$. $\quad f'(x) = \frac{1}{2} \cdot \frac{1}{\cancel{2}}(50^2 + x^2)^{-1/2} \cdot (\cancel{2}x) + (0 - \frac{1}{4})$

② ~~f' DNE~~ $\quad 0 = \frac{x}{2\sqrt{50^2+x^2}} - \frac{1}{4}, \quad \frac{x}{2\sqrt{50^2+x^2}} = \frac{1}{4}$

③ endpoints.

$$x = \frac{1}{2}\sqrt{50^2+x^2}$$

$$x^2 = \frac{1}{4}(50^2 + x^2)$$

$$\frac{3}{4}x^2 = \frac{1}{4}50^2$$

$$3x^2 = 50^2$$

$$x = \frac{50}{\sqrt{3}}$$

x	$f(x)$	
0	$50(3/4)$	local max (= abs max)
$\frac{50}{\sqrt{3}}$		← absolute min.
50	$\frac{50\sqrt{2}}{2}$.	local max

Intuition

$f' > 0 \implies f$ increasing

$f' < 0 \implies f$ decreasing

$f' = 0$ possible local max or min

WW6 # 8. $f(x) = -7x + 4\sin(x)$, $f'(x) = -7 + 4\cos(x) \le -7+4$
≤ -3

Find all intervals where function is increasing

$f' \le -3$ so f always decreasing. NONE intervals of increase.

$(-\infty, \infty)$
(-infinity, infinity) enter into WeBWorK.

Critical points ($f'=0$) since $f' \le -3$, NONE critical points

#9 $f(x) = \dfrac{2x^2}{(x-4)}$ (domain $x \neq 4$)

Determine intervals where f is increasing (and intervals of decrease).

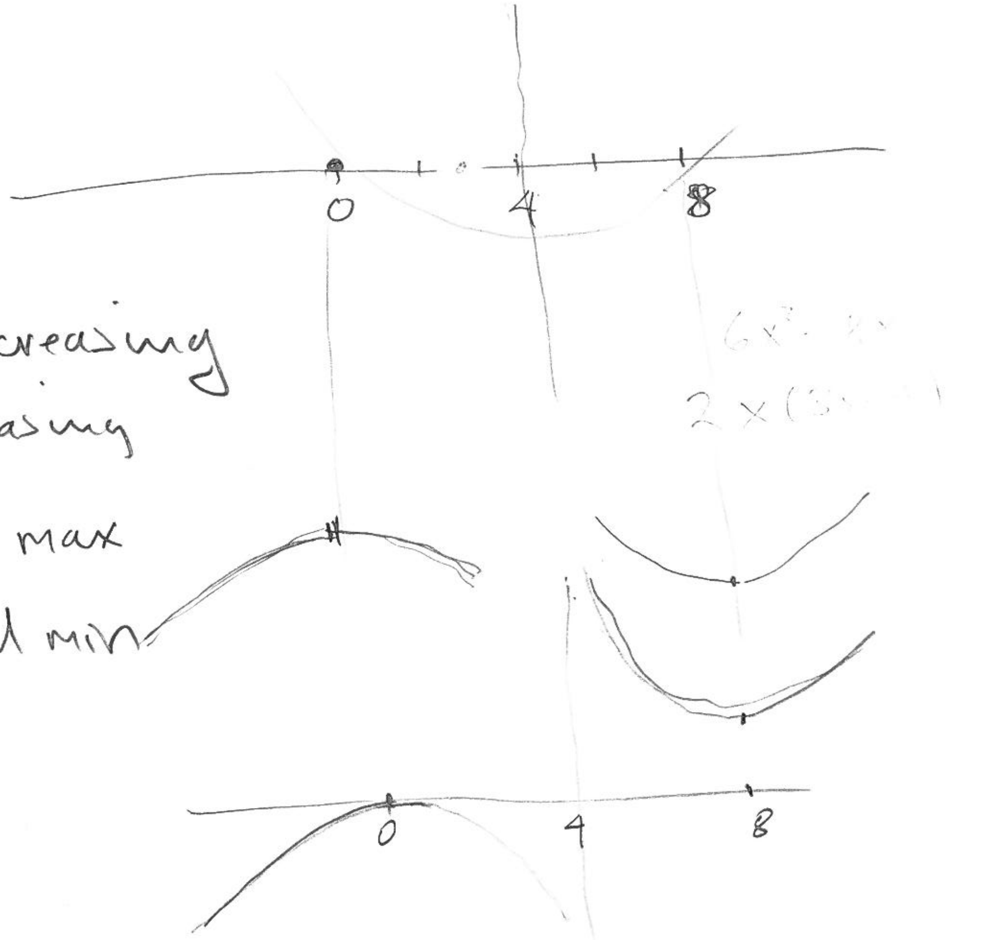
Find critical points ($f' = 0$, ~~DNE~~) f is differentiable on $x \neq 4$.

$$f'(x) = 2 \cdot \frac{2x(x-4) - x^2(1-0)}{(x-4)^2} = 2 \cdot \frac{x^2 - 8x}{(x-4)^2}$$

$2x^2 - 8x - x^2$

$$= 2 \cdot 2 \cdot \frac{x^2 - 8x}{(x-4)^2}$$

$$= 2 \cdot 2 \; \frac{\text{parabola}}{\text{positive}}$$



$f'(x) > 0$ $(-\infty, 0) \cup (8, \infty)$ increasing

$f'(x) < 0$ $(0, 4) \cup (4, 8)$ decreasing

$f'(x) = 0$ at $x = 0, 8$. $\underline{x=0}$ local max

$\underline{x=8}$ local min

#11 Find abs max min values of

$f(x) = (x-2)(x-6)^3 + 5$ differentiable

(a) on closed interval $[1,4]$.

Extreme Value Theorem $\Rightarrow$ abs max / min exist.

Local extreme value theorem $\Longrightarrow$ they are among endpoints, $f'(c)=0$ critical.

$$f'(x) = (x-6)^3 + (x-2)\,3(x-6)^2 \cdot 1$$
$$= (x-6)^2(x-6+3(x-2)) = (x-6)^2(4x-12)$$
$$= 4(x-6)^2(x-3)$$

For interval $[1,4]$, we need check endpoints 1, 4 critical pt 3.

	x	$f(x)$	
abs max	1	130	$(1-2)(1-6)^3+5$ $125+5$
abs min	3	-22	$(3-2)(3-6)^3+5$ $1\cdot(-27)+5$
local max on $[1,4]$	4	-11	$(4-2)(4-6)^3+5$ $2\cdot -8 +5$