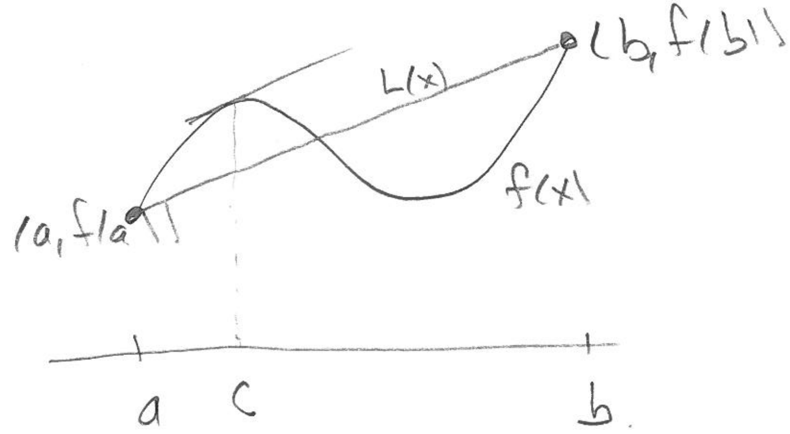
Mean Value Theorem Suppose f is differentiable function on a closed interval $[a,b]$

As we move from one endpoint to the other, at some input c the tangent slope $f'(c)$ equal secant slope $\frac{f(b)-f(a)}{b-a}$



$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Reason: Look at separation between $f(x)$ and secant line $L(x)$

$f(x) - L(x)$. By EVT, there is max/min, and by LEVT, at some interior local max/min c, $(f'(c) - L'(c)) = 0$

$$f'(c) = \text{secant slope} = \frac{f(b)-f(a)}{b-a}$$

Important consequences

1. $f'>0$ on interval $\Rightarrow$ f increasing
   $f'<0$ ——— $\Rightarrow$ f decreasing
2. $f' \equiv 0$ on interval $\Rightarrow$ f is constant

Use ① to justify

<u>1st derivative test</u> If c is a critical point of f, and

1. if $f'>0$ <u>just before c</u>, $f'<0$ <u>just after c</u>, then
   $\Rightarrow$ c yields local max

2. if $f'<0$ <u>just before c</u>, $f'>0$ just after c, then
   $\Rightarrow$ c yield local min

2nd derivative test If c is critical point with $f'(c)=0$

and if

1. $f''(c) > 0$ then f' must increase through c, so f' must switch signs $-$ to $+$
   $\Rightarrow$ c yields local min
2. $f''(c) < 0$ . . . . $\Rightarrow$ c yields local max

Example For $f(x) = e^{(x^3 - x)}$ (domain $(-\infty, \infty)$)

Find critical pt, and use 1st derivative test to determine behavior. Then redo with 2nd derivative test.

Critical pts: $f'(x) = e^{(x^3-x)} \cdot (3x^2 - 1)$.

$$f'(c) = 0 \iff (3c^2 - 1) = 0 \iff c = \pm\sqrt{\tfrac{1}{3}}.$$

Behavior using **1st derivative test**.

Sign of f' determined by $(3x^2 - 1)$

We see $f' > 0$ for $(-\infty, -\sqrt{\tfrac{1}{3}})$, $f' < 0$ for $(-\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}})$

$\Rightarrow$ $-\frac{1}{\sqrt{3}}$ is local max.

At $\sqrt{\tfrac{1}{3}}$ we see f' changes $-$ to $+$, so $\frac{1}{\sqrt{3}}$ is local min

Redo using 2nd derivative test:

$$f''(x) = e^{(x^3-x)}(3x^2-1)\cdot(3x^2-1) + e^{(x^3-x)}\cdot(3\cdot 2x - 0).$$

At critical point $-\frac{1}{\sqrt{3}}$, we get $f''(-\frac{1}{\sqrt{3}}) = \underbrace{e^{(\ )}\cdot 0\cdot 0}_{0} + \underbrace{e^{(\ )}\cdot(6\frac{-1}{\sqrt{3}})}_{\text{neg.}}$

$f''(-\frac{1}{\sqrt{3}})$ is negative

$\Rightarrow$ $-\frac{1}{\sqrt{3}}$ is local max.

Similarly at critical point $\frac{1}{\sqrt{3}}$, we get $f''(\frac{1}{\sqrt{3}}) = 0 + e^{(\ )}(6\frac{1}{\sqrt{3}}) > 0$.

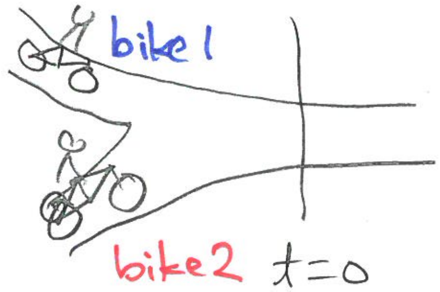
$f''(\frac{1}{\sqrt{3}})$ is positive

$\Rightarrow$ $\frac{1}{\sqrt{3}}$ is local min.

Concavity (Concave up / Concave down)

Idea Use 2nd derivative to tell us something about shape of graph.

Motivation



$b_1(t)$ = position bike 1, $b_2(t)$ = position bike 2

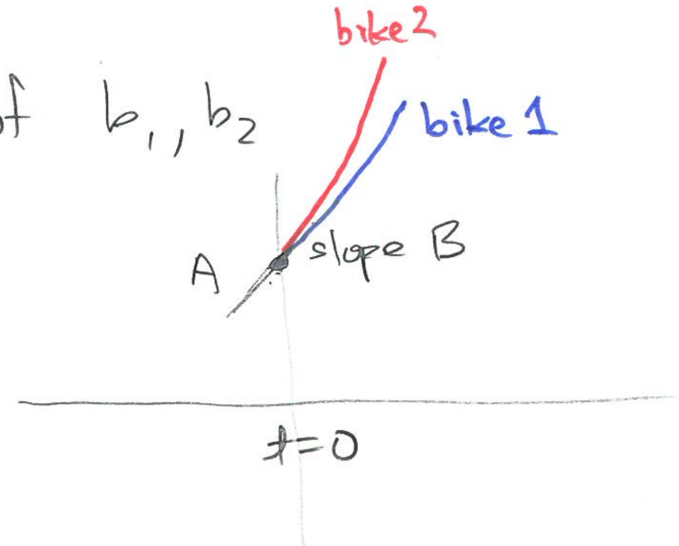
Assume

$A = b_1(0) = b_2(0)$ same starting pt

$B = b_1'(0) = b_2'(0)$ same starting speed.

Assume $b_2''(t) \geq b_1''(t)$

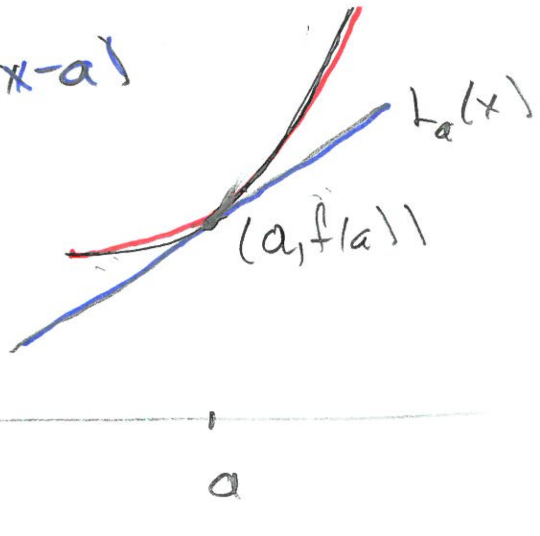
Graph of b_1, b_2



Suppose f is a function and $f'' \geq 0$ near input a.

Let L be the tangent line at a.

$$L_a(x) = f(a) + f'(a)(x-a)$$



$L_a(a) = f(a)$

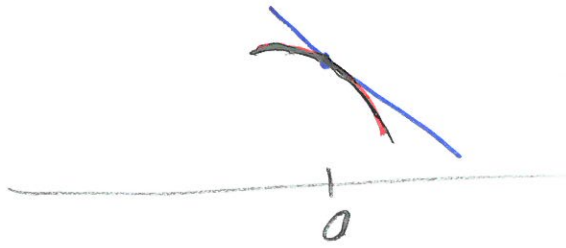
$L_a'(a) = f'(a)$

$L_a''(x) = 0$ since line $\leq$

so $L_a''(x) = 0 \leq f''(x)$

When $f'' \geq 0$ near input a, we graph of f is concave up at input a.

When $f'' \leq 0$ near input a, the graph would look like.



Call this concave down.

If $f'' > 0$ at a, then graph of f is above tangent line near a.

If $f'' < 0$ at a, then graph of f is below tangent line near a.

If graph of f changes from being below to being above tangent line at a (or vice versa) we say a is inflection point

WW6 #12 $\quad f(x) = 2\sin(x) - \frac{\sqrt{3}}{2}x^2$

with domain $[0, 2\pi]$.

There are 2 inflection points. Find them, and determine regions of concave up and concave down.

$f'(x) = 2\cos(x) - \sqrt{3}\,x$, $\quad f''(x) = -2\sin(x) - \sqrt{3}$

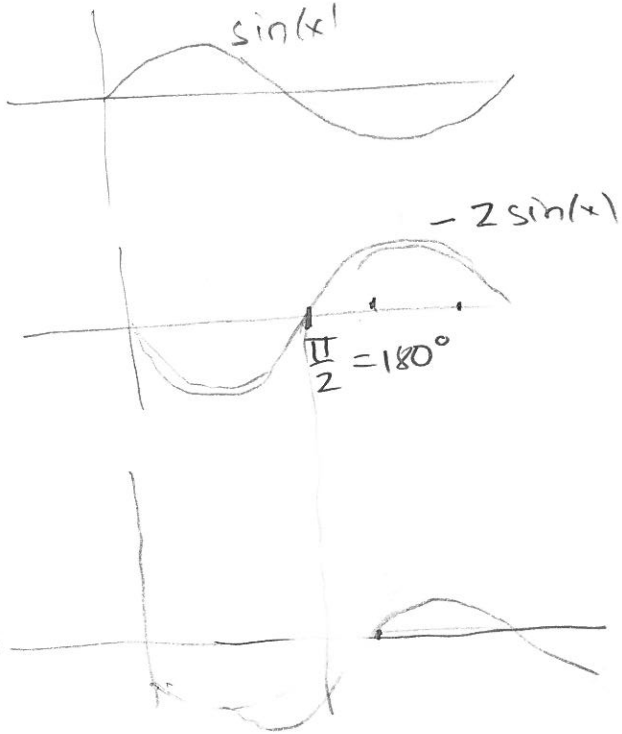
Inflection pts are where $f''(c) = 0$.

Remember $\sin(60) = \frac{\sqrt{3}}{2}$

$f''(c) = 0$ where $\quad 2\sin(c) = -\sqrt{3}$

$\sin(c) = -\frac{\sqrt{3}}{2}$

$$\left.\begin{aligned} c &= \pi + \frac{\pi}{3} = (240^\circ) \\ c &= \pi + \frac{2\pi}{3} = (300^\circ) \end{aligned}\right\} \text{where } f''(c) = 0$$



$[0, \frac{4\pi}{3})$ we see $f'' < 0$, so concave down

$(\frac{4\pi}{3}, \frac{5\pi}{3})$ we have $f'' > 0$, so concave up.

$(\frac{5\pi}{3}, 2\pi]$ we have $f'' < 0$, so concave down