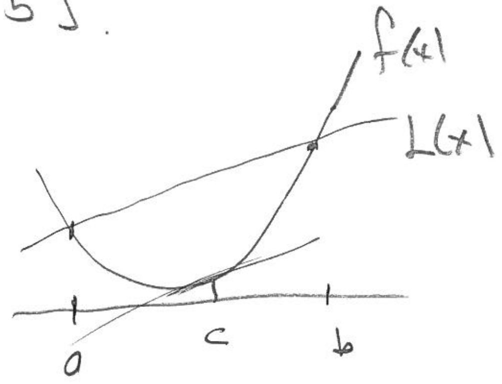
Mean Value Theorem Suppose f is a differentiable function on a closed interval $[a,b]$.



As we move from one endpoint to the other, there is some interior point c so that

$$f'(c) = \text{secant slope between two endpoints}$$
$$= \frac{f(b)-f(a)}{b-a}.$$

Reason. The difference of $f(x) - L(x)$ where $L(x)$ = secant line

it has a max/min (because EVT), and where max/min occurs (say c), we have $f'(c) - L'(c) = 0$ (LEVT)

so $f'(c)$ = slope of secant line $= \dfrac{f(b)-f(a)}{b-a}$.

MVT justifies the intuition

① $f' > 0$ then f increases
$f' < 0$ ——— decreases

② $f' \equiv 0$ on $[a, b]$, the f is constant

We use ① to justify

<u>1st derivative test</u> If c is a critical point of f

(so $f'(c) = 0$, or DNE), then

① if $f' > 0$ just <u>before</u> c, and $f' < 0$ just <u>after</u> c

increasing decreasing

$\Rightarrow$ c yields local max

② if $f' < 0$ just before c, and $f' > 0$ just after c

$\Rightarrow$ c yield local min

2nd derivative test If c is critical point with $f'(c) = 0$.

and if

(1) $f''(c) > 0$ then f' must increase through c so it f' switches from $-$ to $+$ so local min.

(2) $f''(c) < 0$ then f' ——— decrease ———
switches ——— $+$ to $-$ so local max

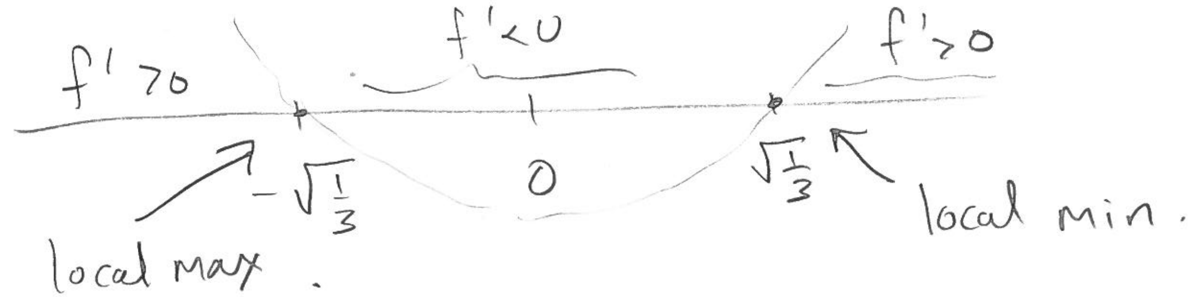
Example. For $f(x) = e^{(x^3 - x)}$ find critical points and use 1st derivative to determine nature. Then redo with 2nd derivative test.

$$f'(x) = e^{(x^3 - x)} \cdot (3x^2 - 1)$$

$f'(c) = 0$ when $3c^2 - 1 = 0$

$c = \pm\sqrt{\frac{1}{3}}$.

1st derivative test



Redo with <u>2nd derivative</u> test

$$f''(x) = e^{(x^3-x)}(3x^2-1)\cdot(3x^2-1) + e^{(x^3-x)}\cdot(3\cdot 2x - 0)$$

Values of f'' at critical pts.

$$f''\left(\frac{-1}{\sqrt{3}}\right) = e^{(\quad)}(0)\cdot(0) + e^{(\quad)}\cdot\left(6\left(\frac{-1}{\sqrt{3}}\right)\right) = 0 + \text{neg} < 0$$

so local <u>max</u>

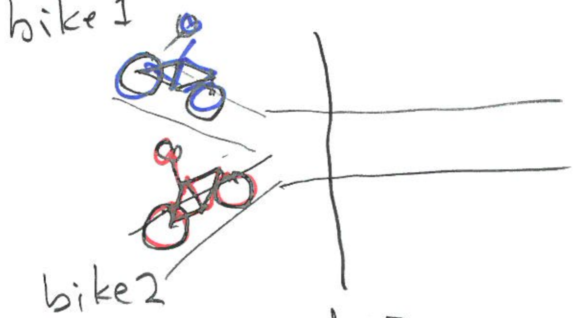
$$f''\left(\frac{1}{\sqrt{3}}\right) = e^{(\quad)}(0)(0) + e^{(\quad)}\cdot\left(6\left(\frac{1}{\sqrt{3}}\right)\right) > 0$$

so local min.

Concavity (concave up / concave down)

Idea The sign of 2nd derivative tells us about shape of graph.

Motivation



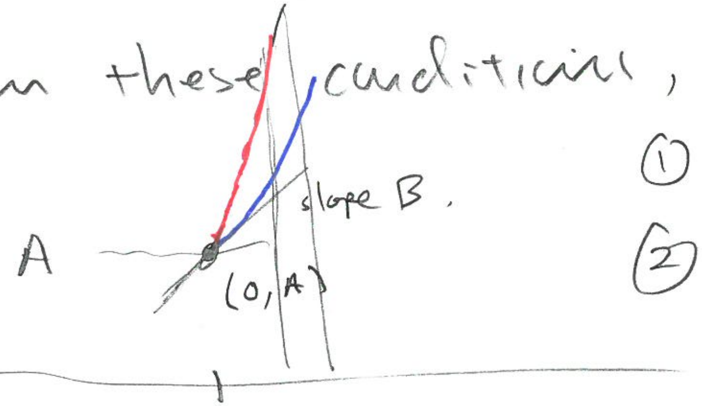
Suppose at $t=0$

$b_1(0) = A$, $b_2(0) = A$

$b_1'(0) = B$, $b_2'(0) = B$

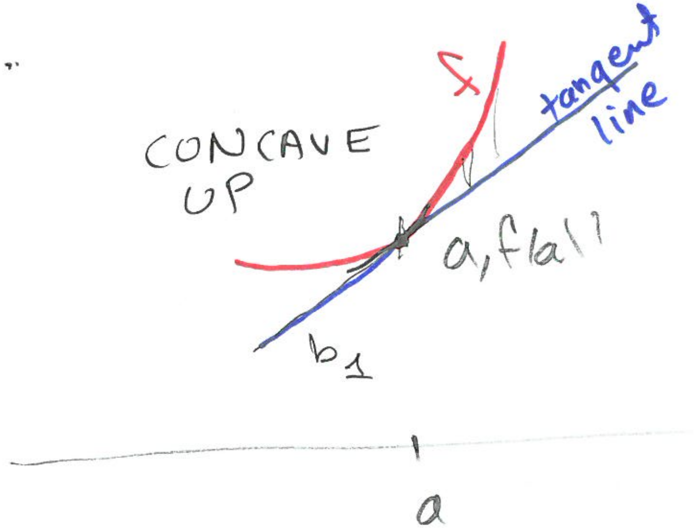
Assume $b_2''(t) \geq b_1''(t)$

Intuition. Given these conditions, then



1. $b_2(t) \geq b_1(t)$ for $t > 0$
2. separation between b_2 and b_1 increases.

Suppose $f(x)$ is a function, and $f'' \geq 0$ around the point a.



We consider

$$b_2(x) = f(x)$$

$$b_1(x) = f(a) + f'(a)(x-a)$$

$$b_2(a) = f(a) = b_1(a)$$

$$b_2'(a) = f'(a) = b_1'(a)$$

2nd derivative of line

$$b_1'' = 0$$

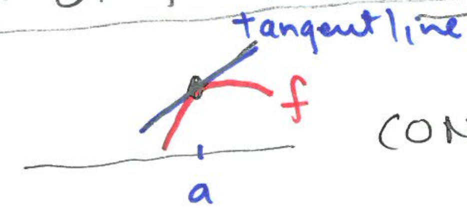
2nd derivative of b_2 is

$$b_2'' = f'' \geq 0 = b_1''.$$

Conclusion: If $f'' > 0$ around point a, the graph of f is above tangent line at $(a, f(a))$ and increases as we go away from a.

We say the graph of f is concave up around a.

If $f'' < 0$ then



CONCAVE DOWN.

If c is a point where the 2nd derivative switches from $+$ to $-$ or $-$ to $+$ we say c is an <u>inflection</u> point.

At such a point the graph crosses the tangent line.

WW6 #12 $\quad f(x) = 2\sin(x) - \frac{\sqrt{3}}{2}x^2$

use $[0, 2\pi]$ as domain. Find 2 inflection points

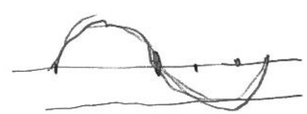
So find where f'' switches signs.

$$f'(x) = 2\cos(x) - \frac{\sqrt{3}}{2}\cdot 2x = 2\cos(x) - \sqrt{3}\,x$$

$$f''(x) = -2\sin(x) - \sqrt{3}. \quad (f'' \text{ is continuous})$$

Where f'' changes sign is where f'' is 0.

sin(x)



$$0 = -2\sin(x) - \sqrt{3} \quad \text{so} \quad 2\sin(x) = -\sqrt{3}$$

$$\sin(x) = -\frac{\sqrt{3}}{2}$$

$$x = \left(\pi + \frac{\pi}{3}\right) \text{ or } \left(\pi + \frac{2\pi}{3}\right)$$

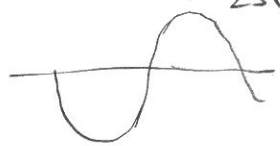
240° $\qquad$ 300°.

180+60

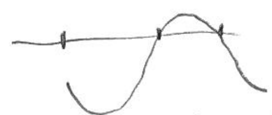
240

- $[0, \frac{4\pi}{3})$ has $f'' < 0$ so concave down

-2sin(x)



- $(\frac{4\pi}{3}, \frac{5\pi}{3})$ has $f'' > 0$ so concave up
- $(\frac{5\pi}{3}, 2\pi]$ has $f'' < 0$ so concave down.



-2sin(x)-√3