<u>Concavity</u> (concave up, concave down). Assume f is a function which has 1st and 2nd derivatives.

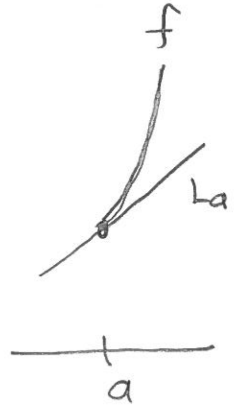
Suppose near input a, the 2nd derivative $f'' > 0$

We compare f to its tangent line a input a

$$L_a(x) = f(a) + f'(a)(x-a).$$

So f, and L_a have (i) same starting point at a both values are $f(a)$

(ii) same starting speed at a both values are $f'(a)$.



(iii) L_a has $L_a'' = 0$ ("no acceleration"), while $f'' > 0$

Let $G_a(x) = f(x) - L_a(x)$ = gap between the two functions

So $G_a(a) = 0$, $G_a'(a) = 0$, $G_a'' = f'' - L_a'' = f'' > 0$ near a.

Since $(G_a')' > 0 \Rightarrow$ so G_a' increasing ($G_a'(a) = 0$) so $G_a' > 0$ for $x > a$.

Since $G_a' > 0$ for $x > a \Rightarrow G_a$ increasing ($G_a(a) = 0$)

So for values $x > a$ (near a), graph of f is above tangent line.

Can argue same for x below a.

Clever way is:

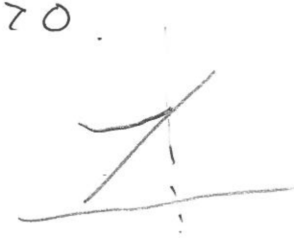
Look at $f(2a-x) = g(x)$

reflection.

$$g'(x) = f'(2a-x)(-1)$$

and $g''(x) = f''(2a-x)(-1)(-1) = f''(2a-x) > 0$.

This means going negative direction same



So when $f''>0$ near a, graph of f

is above tangent line L_a.

Similarly $f''<0$ near $a \Rightarrow$ graph below tangent line

Example $f(x) = e^{-\frac{x^2}{2\sigma^2}}$ (Gaussian function).

Analyze this function

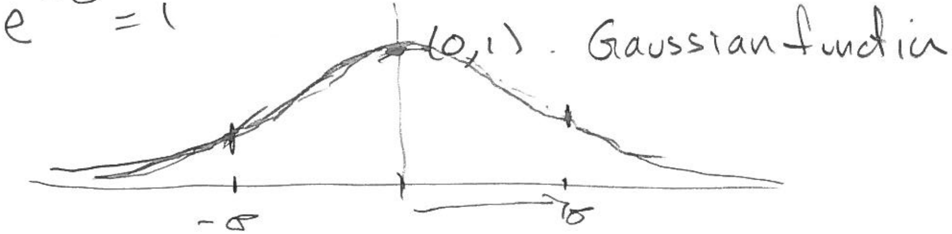
(i) asymptotes

(ii) local max/min

(iii) concavity

Note f is EVEN $f(-x) = f(x)$.

$f(0) = e^{-0} = 1$



Asymptotes: No vertical asymptote.

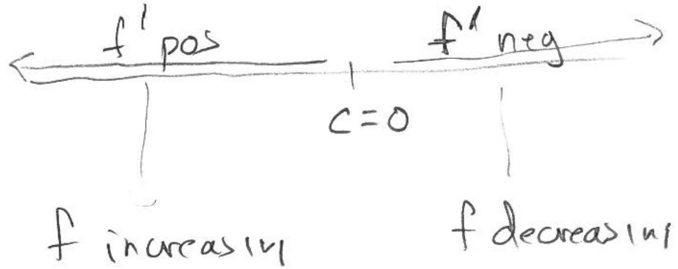
As $x \to +\infty$ then $f(x) \to 0^+$ so $y=0$ is horizontal asymptote

$x \to -\infty$ $\quad f(x) \to 0^+$

Local max/min: f' exists everywhere. Critical pt is where $f' = 0$.

$$f'(x) = \left(e^{-\frac{x^2}{2\sigma^2}}\right)\left(-\frac{2x}{2\sigma^2}\right) = -\left(e^{-\frac{x^2}{2\sigma^2}}\right)\left(\frac{x}{\sigma^2}\right).$$

so $f'(c) = 0$ when $c = 0$, and



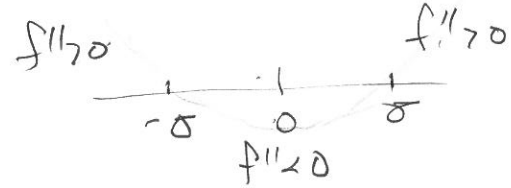
Since f' changes $+$ to $-$ at $c=0$,

$c=0$ is a local max.

Concavity: $f''(x) = -\left(\left(e^{-\frac{x^2}{2\sigma^2}}\right)\cdot\left(\frac{x}{\sigma^2}\right)\right)'$

$$= -\left(-\left(e^{-\frac{x^2}{2\sigma^2}}\right)\left(\frac{x}{\sigma^2}\right)\cdot\left(\frac{x}{\sigma^2}\right) + \left(e^{\frac{-x^2}{2\sigma^2}}\right)\left(\frac{1}{\sigma^2}\right)\right)$$

$$= \underbrace{e^{\left(-\frac{x^2}{2\sigma^2}\right)}\left(\frac{1}{\sigma^2}\right)}_{\text{positive}}\left(\left(\frac{x^2}{\sigma^2}\right)-1\right)$$



Concavity determined by <u>sign</u> of f''. Sign f'' same as sign of $\left(\frac{x^2}{\sigma^2}\right)-1$

So $f'' > 0$ on $(-\infty, -\sigma) \cup (\sigma, \infty)$ concave up

$f'' < 0$ on $(-\sigma, \sigma)$ concave down

f'' changes sign at $-\sigma$ and σ. These are inflection points.

In statistics, the inflection point σ is called <u>standard deviation</u>

WW6 #14 Analyze the function $f(x) = 7x^2 \ln(x)$ domain $x > 0$.
and use your information to draw rough graph

(a) Critical points/numbers: $f'(x) = 7 \cdot \left(2x \ln(x) + x^2 \frac{1}{x}\right)$

$$= 7 \cdot (2x \ln(x) + x)$$

$$= 7x\,(2\ln(x) + 1) \qquad \underline{x > 0}$$

$f'(c) = 0$ and $x > 0$

$2\ln(c) + 1 = 0$, $c = e^{-1/2}$

$f' < 0$ $\qquad$ $f' > 0$

0 $\qquad e^{-3/2}$ $\qquad e^{-1/2}$ $\qquad 1$

f decreasing $(0, e^{-1/2})$ $\qquad$ f increasing $(e^{-1/2}, \infty)$

(b) f increasing $(e^{-1/2}, \infty)$

(c) f decreasing $(0, e^{-1/2})$

(d) NO local max

(e) $e^{-1/2}$ yields local min

$f(1) = 7 \cdot 1^2 \ln(1) = 0$.

Concavity. $f''(x) = (7x(2\ln(x)+1))' = 7\left(1 \cdot (2\ln(x)+1) + x \cdot \left(\frac{2}{x}\right)\right)$

$$= 7 \cdot (2\ln(x) + 1 + 2) = 7 \cdot (2\ln(x) + 3).$$

$f''(d) = 0$ where

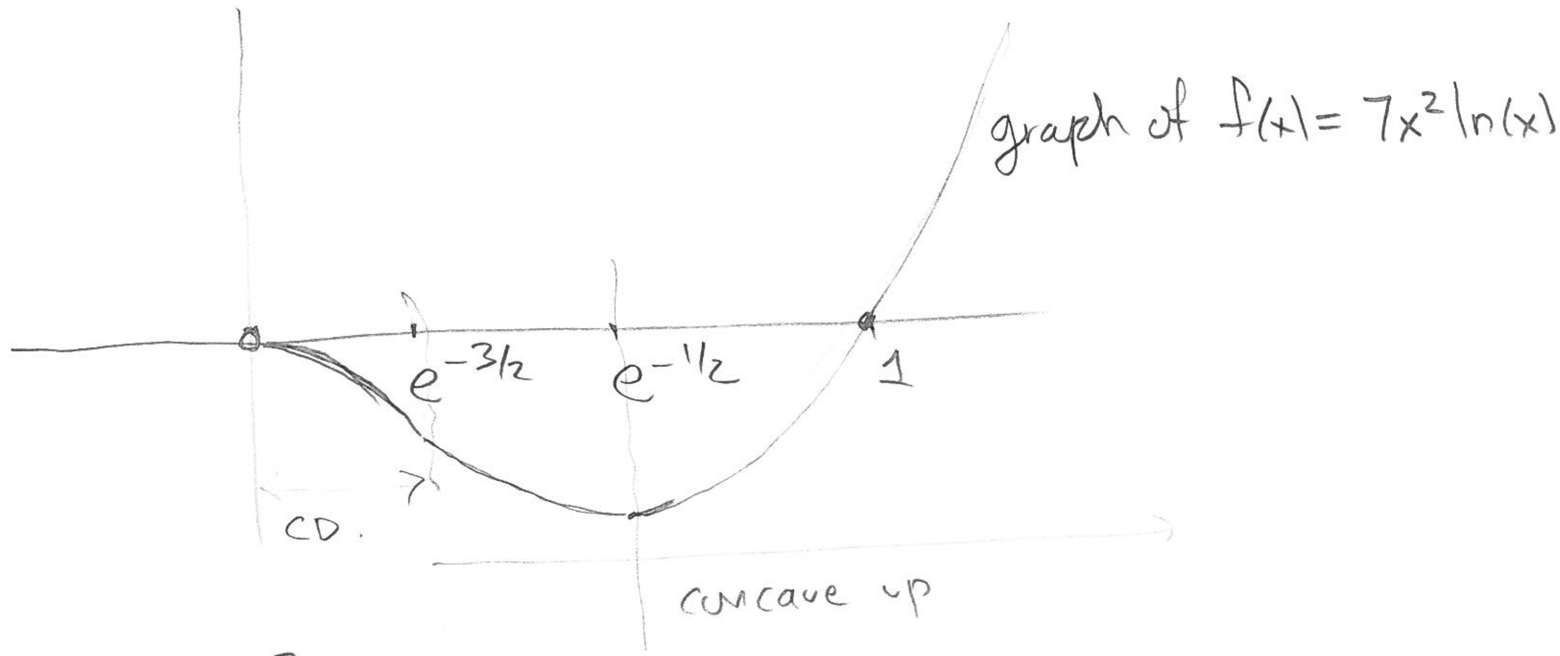
$2\ln(d) + 3 = 0$, $d = e^{-3/2}$

$f'' < 0$ $\qquad$ $f'' > 0$ concave up

0 concave down $\qquad e^{-3/2}$ $\qquad 1$

inflection point

$f(1) = 0$



What happens as $x \to 0^+$?

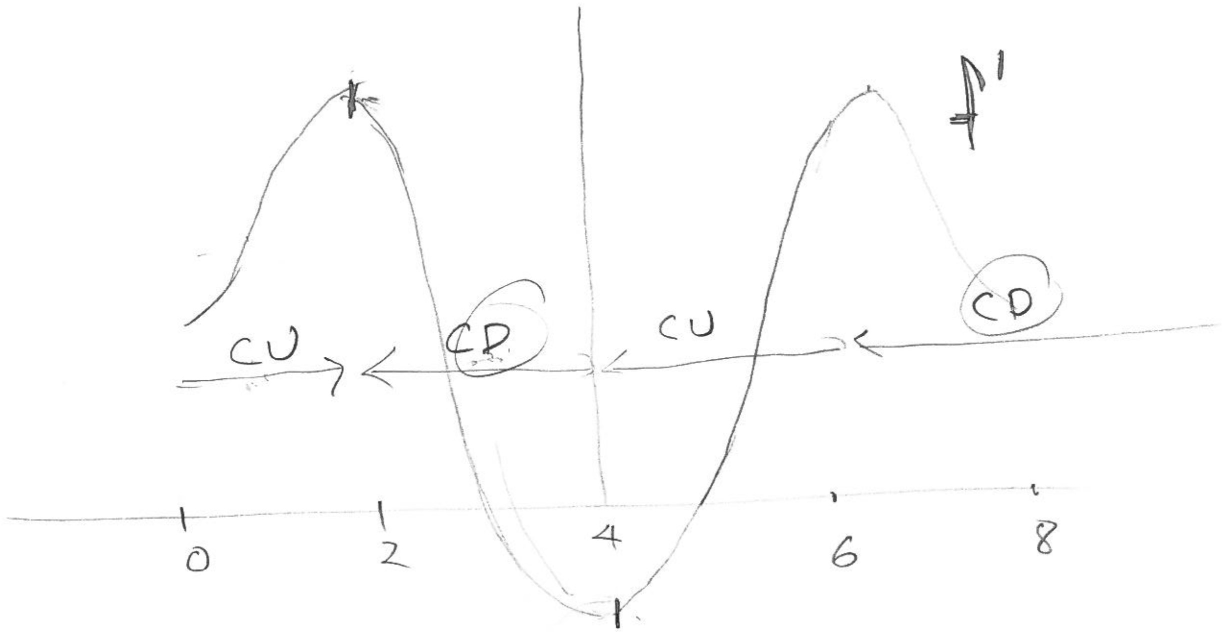
$$\lim_{x \to 0^+} 7x^2 \ln(x) = \lim_{u \to -\infty} 7\, e^{2u}\, u = 7 \cdot 0$$

$x = e^u$, $u = \ln(x)$

exponentials alway stronger than polynomials

So as $x \to 0^+$, we have $f(x) \to 0^-$

#16 Graph of DERIVATIVE



(a) Where is f concave down? Need $\underline{\underline{f'' < 0}} \Rightarrow f'$ decreasing

$(2,4) \cup (6,8)$

(b) inflection point 2, 4, 6.