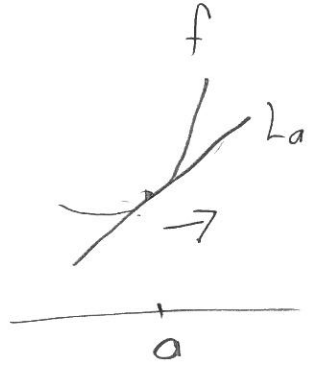
Concavity (Up, Down)

Suppose f is function on an interval with 1st and 2nd derivatives and we assume $f''>0$ near the input a.

We compare f with its tangent line L_a (at input a)

$$L_a(x) = f(a) + f'(a)(x-a)$$



The two functions f, and L_a, at input a.

(i) have same value
(ii) have same derivative value

Note $L_a''(x) = 0$ (2nd derivative of line is 0).

Using assumption $f''>0$ near a.

Consider "gap" $f(x) - L_a(x)$. So

$(f-L_a)(a) = 0$
$(f-L_a)'(a) = 0$
$(f-L_a)''(a) > 0$

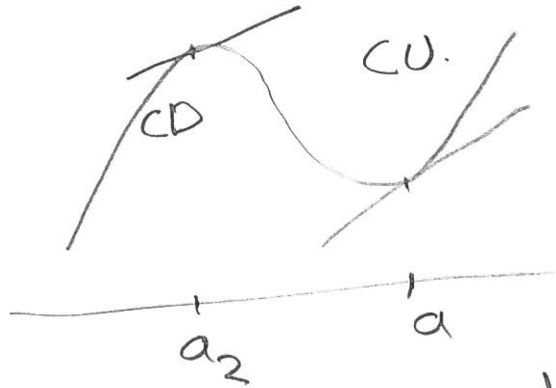
$(f-L_a)'' > 0 \Rightarrow (f-L_a)'$ is increasing

Since $(f-L_a)'(a) = 0$, $\Rightarrow (f-L_a)'(x) > 0$ for $x > a$.

so $(f-L_a)$ increasing (so gap is increasing)

Conclude as we move away from a in + direction that graph of f is above tangent line and the gap increases.

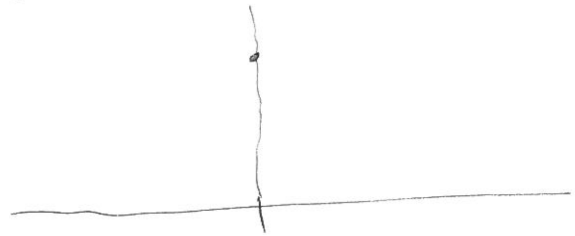
We call this concave up (CU)



If $f'' < 0$ near point a_2, then graph of f is below tangent line (concave down) CD

Example $f(x) = e^{-\frac{x^2}{2\sigma^2}}$ domain $(-\infty,\infty)$

even function



Analyze this function

(i) asymptotes

(ii) local max/min

(iii) concavity.

(i) No vertical asymptotes.

As $x \to +\infty$, then $x^2 \to +\infty$, so $-\frac{x^2}{2\sigma^2} \to -\infty$, so $e^{-\frac{x^2}{2\sigma^2}} \to 0$.

$y=0$ is horizontal asymptote.

Similarly as $x \to -\infty$.

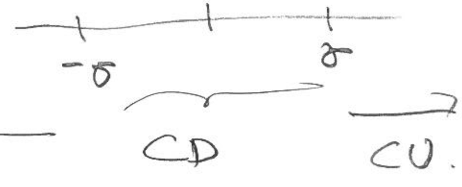
(ii) $f'(x) = \left(e^{-\frac{x^2}{2\sigma^2}}\right)\left(-\cancel{2}x\left(\frac{1}{\cancel{2}\sigma^2}\right)\right) = -e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x}{\sigma^2}\right)$

f' has same sign as $-\frac{x}{\sigma^2}$. so f increasing $(-\infty,0)$, decreasing $(0,\infty)$, $f'(0)=0$ } so in pt $x=0$ is local max

(iii) Concavity $f''(x) = -\left\{ -e^{-\frac{x^2}{2\sigma^2}}\left(\frac{x}{\sigma^2}\right)\left(\frac{x}{\sigma^2}\right) + e^{-\frac{x^2}{2\sigma^2}}\left(\frac{1}{\sigma^2}\right)\right\}$

$$= \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sigma^2}\left\{\frac{x^2}{\sigma^2}-1\right\}.$$

Sign of 2nd derivative is same as of $\left(\frac{x^2}{\sigma^2}-1\right)$.

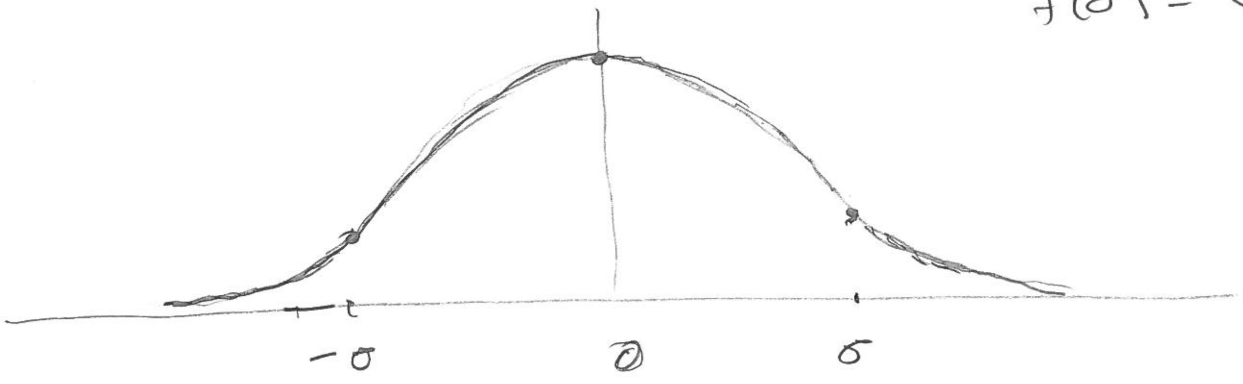


when $x^2 > \sigma^2$ (so $x > \sigma$, or $x < -\sigma$)
then $f'' > 0$ so concave up.

when $x^2 < \sigma^2$ so $-\sigma < x < \sigma$, then $f'' < 0$ so concave down

$f(\sigma) = e^{-\frac{\sigma^2}{2\sigma^2}} = e^{-1/2}$

Graph



Since f'' changes sign at $-\sigma$, and σ, these are <u>inflection points</u>

The inflection points σ (and $-\sigma$) is called the standard deviation.

WW6 # 14 Analyze function

$f(x) = 7x^2 \ln(x)$ domain $\underline{\underline{x>0}}$

(a) Find critical points. Since f differentiable on domain. Critical pts are where $f'=0$.

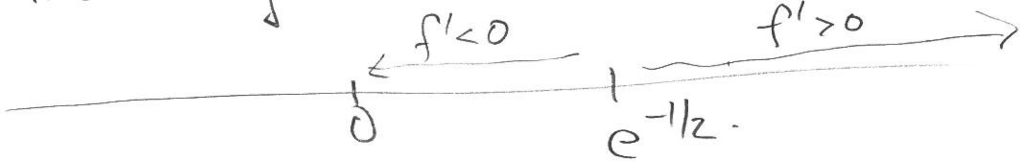
$$f'(x) = 7\cdot\left\{2x\cdot\ln(x) + x^2\,\frac{1}{x}\right\} = 7\left\{2x\cdot\ln(x) + x\right\}$$

$$= 7x\left\{2\ln(x)+1\right\}.$$

$f'=0$ on domain $x>0$ only at point $2\ln(x)+1=0$

$c = e^{-1/2}$ ← yields local min.

(b) where is f increasing? Where $f'>0$



increasing in interval $(e^{-1/2}, \infty)$

(c) where is f decreasing? Where $f'<0$, so $(0, e^{-1/2})$

(d) Are there any local $\underline{\text{MAX}}$? NO (NONE)

(e) Are there any local MIN? Yes $e^{-1/2}$.

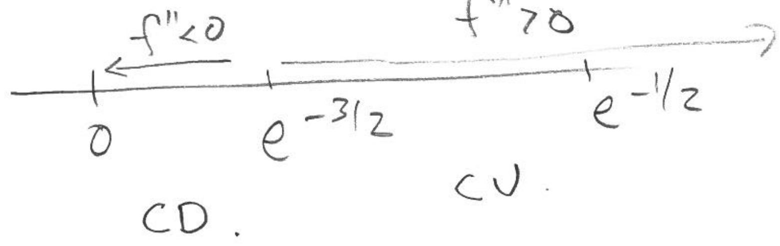
(f), (g) Where is f concave up, where concave down?

Since $f'(x) = 7x\{2\ln(x)+1\}$, we have

$$f''(x) = 7\cdot\{1(2\ln(x)+1) + x(2\tfrac{1}{x}+0)\} \qquad 1+2=3$$

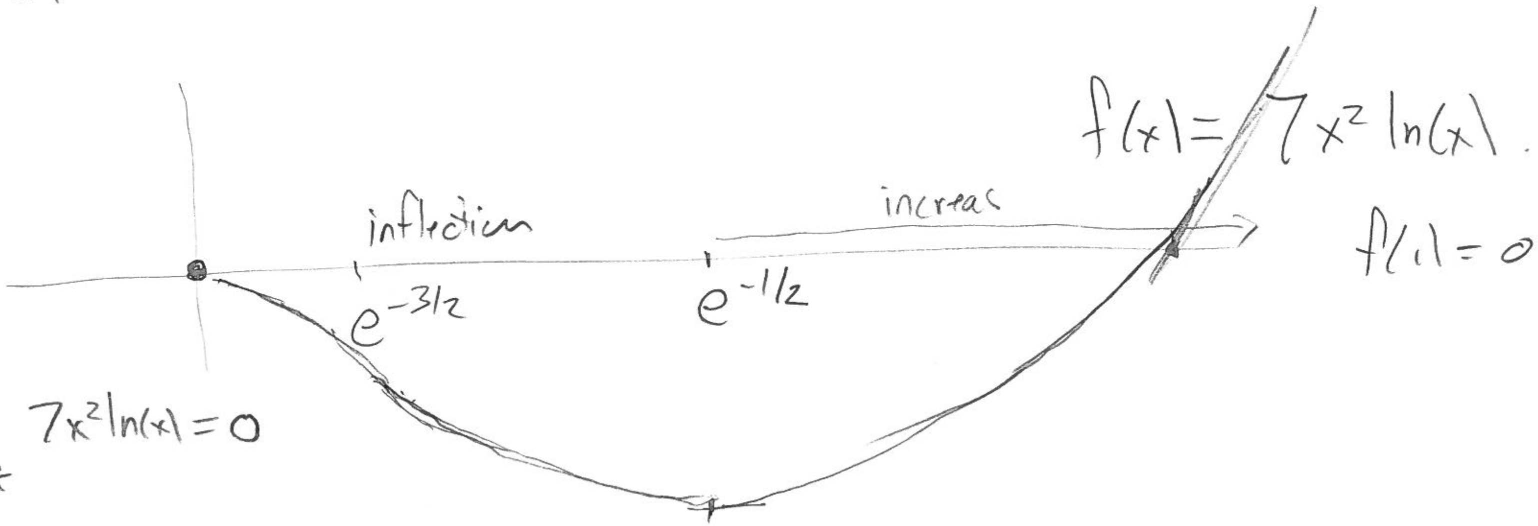
$$= 7\cdot\{2\ln(x)+3\}.$$

$f''=0$ when $2\ln(d)+3=0$ $\quad \ln(d)=-3/2$, $\quad d=e^{-3/2}$



(h) inflection point at $e^{-3/2}$

Sketch graph.



Since $\lim_{x\to 0^+} 7x^2\ln(x) = 0$

Domain $x>0$.. What happens in $(0, e^{-3/2})$ concave down.

$\lim_{x \to 0^+} 7x^2 \ln(x)$ = indeterminant $7 \cdot 0 \cdot (-\infty)$.

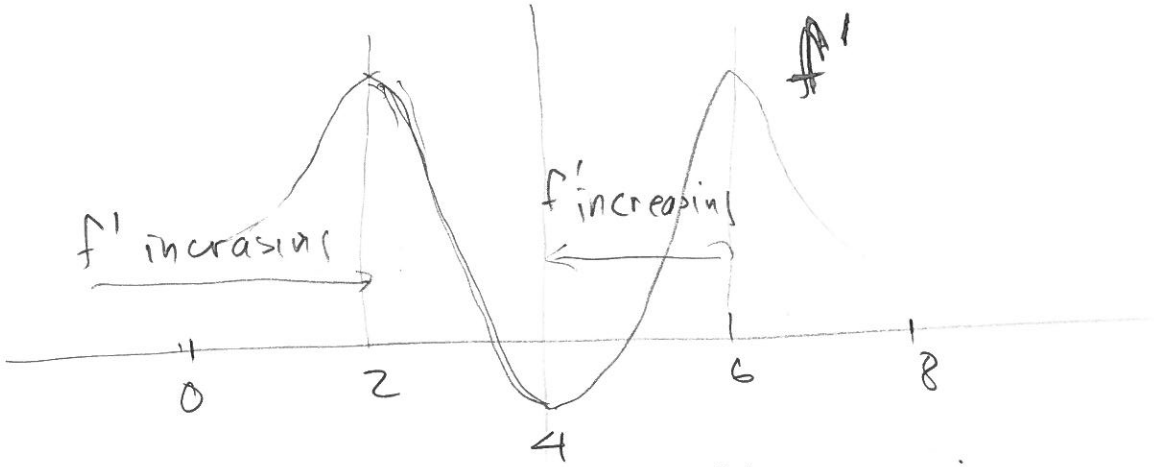
We manipulate $7x^2 \ln(x)$, we set $x = e^u$ so

$7x^2 \ln(x) = 7e^{2u} \ln(e^u) = 7e^{2u}u.$

| $x \to 0^+$, $u \to -\infty$.

$\lim_{u \to -\infty} 7(e^{2u}u) = 7 \cdot 0 = 0$. So $\lim_{x \to 0^+} 7x^2 \ln(x) = 0$.

#16 Graph of DERIVATIVE



concave up is $f'' > 0 \Rightarrow f'$ increasing $(0,2) \cup (4,6)$

concave down is $f'' < 0 \Rightarrow f'$ decreasing $(2,4) \cup (6,8)$

2, 4, 6 are where concavity changes $\text{UP}\underbrace{,}_{2}\text{ DOWN}\underbrace{,}_{4}\text{ UP}\underbrace{,}_{6}\text{ DOWN}$