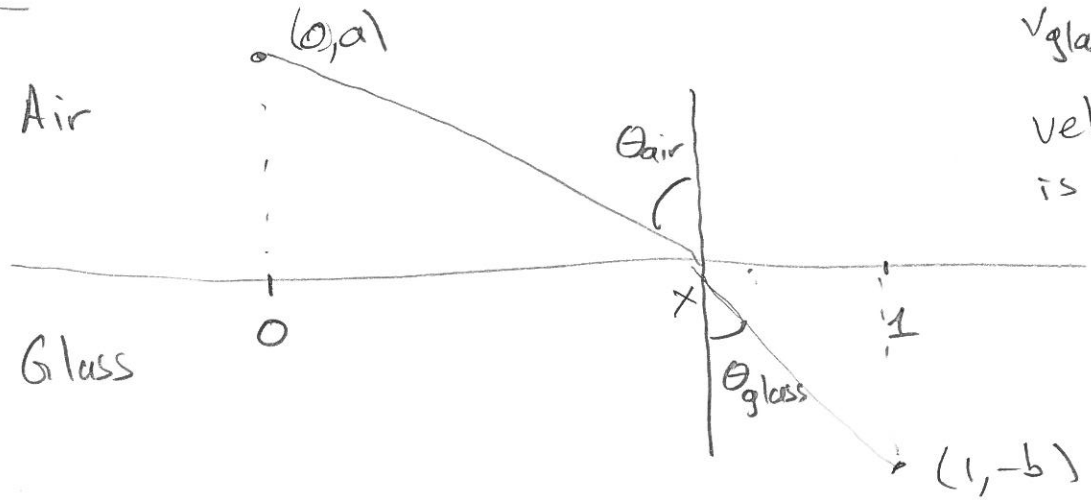
Week 10 Friday L03 2pm

Snell's Law

$$\frac{v_{air}}{v_{glass}} = 1.5 = \frac{3}{2}$$

velocity of light in air is 1.5 that in glass.

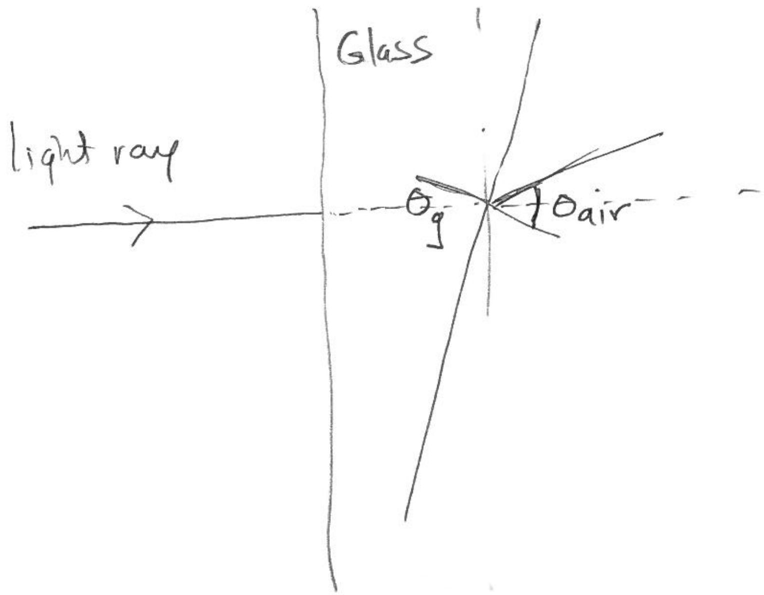


Light finds shortest travel time

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_{air}} + \frac{\sqrt{b^2 + (1-x)^2}}{v_{glass}}$$ Find min.

Take derivative to find critial pt.

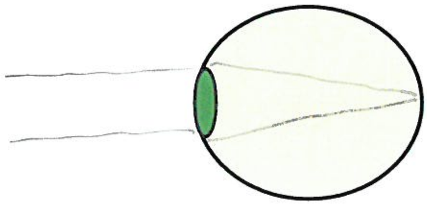
$$\frac{3}{2} = \frac{v_{air}}{v_{glass}} = \frac{\sin(\theta_{air})}{\sin(\theta_{glass})}$$



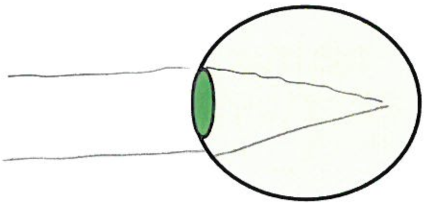
$$\frac{\sin(\theta_{air})}{\sin(\theta_{glass})} = \frac{3}{2}$$

$$\sin(\theta_{air}) = \frac{3}{2}\sin(\theta_{glass})$$

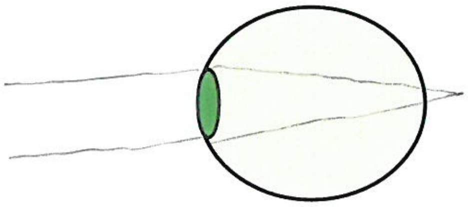
$$\theta_{air} = \arcsin\left(\frac{3}{2}\sin(\theta_{glass})\right)$$



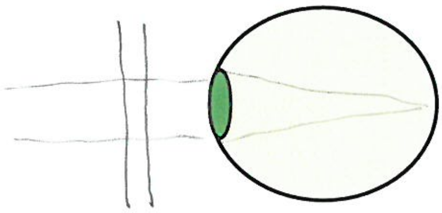
Normal



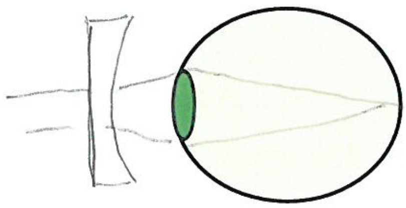
Nearsighted



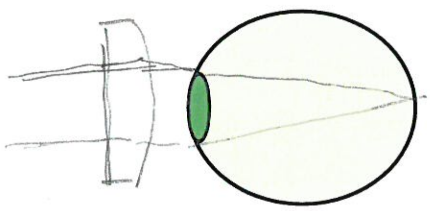
Farsighted



Normal



Nearsighted



Farsighted

Limits and L'hopital's Rule

Example Find $\lim_{x \to 1} \frac{\ln(x)}{x^2-1}$

$f(x) = \ln(x)$ is continuous
As $x \to 1$, $f(x) \to f(1) = \ln(1) = 0$

$g(x) = x^2 - 1$ is continuous
As $x \to 1$, $g(x) \to g(1) = 1^2 - 1 = 0$.

$\frac{0}{0}$ indeterminant (have to be more careful).

$$\frac{\ln(x)}{x^2-1} = \frac{f(x)}{g(x)} = \frac{f(x)-f(1)}{g(x)-g(1)}$$

$$= \frac{\frac{f(x)-f(1)}{x-1}}{\frac{g(x)-g(1)}{x-1}}$$

As $x \to 1$ $\frac{\text{top} \to \text{tangent slope of } f \text{ at input } 1}{\text{bottom} \to \text{'' '' '' } g \text{ at input } 1}$

$f(x) = \ln(x)$ so $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.
$g(x) = x^2 - 1$ so $g'(x) = 2x$ and $g'(1) = 2$

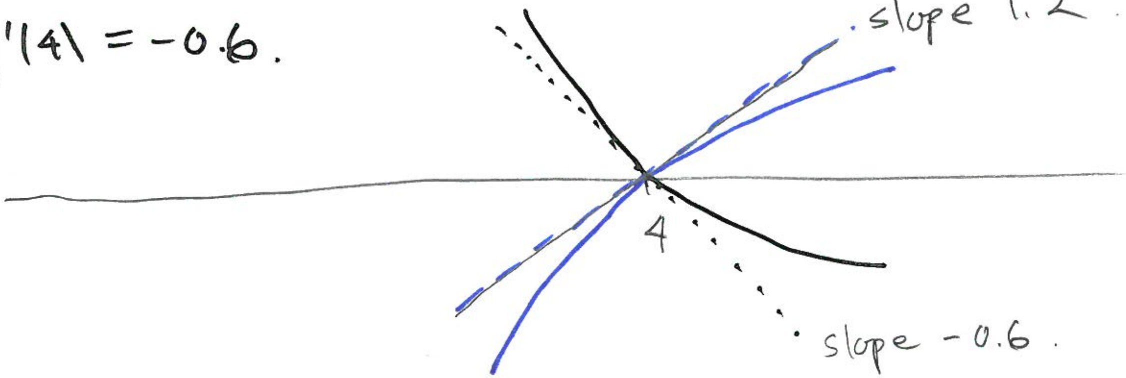
$$= \frac{1}{2}$$

So $\lim_{x \to 1} \frac{\ln(x)}{x^2-1} = \frac{1}{2}$

WW7 #5 Two functions

$f(4)=0,\ f'(4)=1.2$

$g(4)=0,\ g'(4)=-0.6.$



Find $\lim_{x\to 4} \frac{f(x)}{g(x)}$

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(4)}{g(x)-g(4)} = \frac{\left(\frac{f(x)-f(4)}{x-4}\right)}{\left(\frac{g(x)-g(4)}{x-4}\right)}$$

As $x \to 4$ $\dfrac{f \text{ secant slope} \longrightarrow \text{tangent slope } 1.2}{g \text{ secant slope} \longrightarrow \text{tangent slope } -0.6}$

Therefore $\lim_{x\to 4} \frac{f(x)}{g(x)} = \frac{1.2}{-0.6} = -2.$

L'hopital's Rule Suppose $\lim_{x \to a} \frac{f(x)}{g(x)}$ satisfies

① $\lim_{x \to a} f(x) = 0$ AND $\lim_{x \to a} g(x) = 0$.

so $\frac{0}{0}$ indeterminant

② $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ too.

WW7 #6 Find $\lim_{x \to 0} \frac{x^3}{\sin x - x}$

$f(x) = x^3$, $g(x) = \sin x - x$ both continuous

As $x \to 0$, $f(x) \to f(0) = 0$

$g(x) \to g(0) = 0 - 0 = 0$.

$\frac{0}{0}$ indeterminant

Look a derivative ratio: $\frac{f'(x)}{g'(x)} = \frac{3x^2}{\cos x - 1}$ These also continuous

As $x \to 0$, $f'(x) \to 3 \cdot 0^2 = 0$

This also indeterminant $\frac{0}{0}$. , $g'(x) \to \cos 0 - 1 = 1 - 1 = 0$

Try L'hopital again

$\frac{f''(x)}{g''(x)} = \frac{3 \cdot 2 \cdot x}{-\sin x}$ Again continuous top/bottom

As $x \to 0$, $f''(x) \to f''(0) = 0$ $\frac{0}{0}$.

$g''(x) \to g''(0) = 0$

One more time

$\frac{f'''(x)}{g'''(x)} = \frac{3 \cdot 2 \cdot 1}{-\cos x} \to \frac{6}{-1}$

As $x \to 0$ so $\lim_{x \to 0} \frac{f(x)}{g(x)} = -6$.

#7 Find $\lim_{x \to 3} \frac{\ln(x/3)}{x^2 - 9}$.

Could do this with derivative like #5.

Instead we use L'hopital's rule.

$$f(x) = \ln(x/3), \text{ continuous} \quad f(3) = \ln(3/3) = 0.$$

verify indeterminant

$$g(x) = x^2 - 9, \quad \text{"} \quad \text{"} \quad g(3) = 3^2 - 9 = 0. \quad \frac{0}{0}.$$

$$\frac{f'(x)}{g'(x)} \begin{matrix} = (\ln(x) - \ln(3))' \\ = 2x - 0 \end{matrix} = \frac{\frac{1}{x}}{2x}$$

As $x \to 3$ $\begin{cases} \frac{1}{x} \to \frac{1}{3} \\ 2x \to 6 \end{cases}$ so $\frac{f'(x)}{g'(x)} \to \frac{\frac{1}{3}}{6} = \frac{1}{18}$

WARNING. When applying l'hopital's rule, you <u>must</u> verify $\frac{0}{0}$ indeterminant

Example Find $\lim_{x \to \frac{\pi}{3}} \frac{\left(\frac{1}{2} - \cos(x)\right)}{\sin x}$

$f(x) = \frac{1}{2} - \cos(x)$ continuous

$f\left(\frac{\pi}{3}\right) = \frac{1}{2} - \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} - \frac{1}{2} = 0$

$f'(x) = 0 + \sin(x)$

$g'(x) = \cos(x)$

$\frac{f'(x)}{g'(x)} = \frac{\sin(x)}{\cos(x)} \longrightarrow \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$ as $x \to \frac{\pi}{3}$.

Note $g(x) = \sin(x)$ (continuous) $\lim_{x \to \frac{\pi}{3}} \sin(x) = \frac{\sqrt{3}}{2}$.

So $\lim_{x \to \frac{\pi}{3}} \frac{\left(\frac{1}{2} - \cos(x)\right)}{\sin x} = \frac{0}{\left(\frac{\sqrt{3}}{2}\right)} = 0$ (NOT)

#10 f continuous and satisfies

$f(7) = 0, \quad f'(7) = 14$

$h(x) = f(7+3x) + f(7+5x)$ top function

Find $\lim_{x \to 0} \dfrac{f(7+3x) + f(7+5x)}{x} = 14 \cdot 8$. Set $g(x) = x$ bottom function

- As $x \to 0$, we have $7+3x \to 7+3 \cdot 0 = 7$ so $f(7+3x) \to f(7) = 0$

$7+5x \to 7+3 \cdot 0 = 7$ so $f(7+5x) \to f(7) = 0$

So $f(7+3x) + f(7+5x) \longrightarrow 0 + 0 = 0$.

- As $x \to 0$, the bottom function x has limit 0.

So we do have $\frac{0}{0}$ indeterminant. Apply L'hopital

$$\frac{h'(x)}{g'(x)} = \frac{f'(7+3x) \cdot (0+3) + f'(7+5x) \cdot (0+5)}{1}$$

As $x \to 0$, we see

$h'(x) \Rightarrow f'(7+3 \cdot 0) \cdot 3 + f'(7+5 \cdot 0) \cdot 5$

$= f'(7) \cdot 3 + f'(7) \cdot 5$

$= 14 \cdot 3 + 14 \cdot 5$

$$\lim_{x \to 0} \frac{h'(x)}{g'(x)} = \frac{14 \cdot 3 + 14 \cdot 5}{1} = 14 \cdot 8$$

Other versions of L'hopital's rule.

If ① $\lim_{x \to a} \frac{f(x)}{g(x)}$ has indeterminiant form $\frac{\infty}{\infty}$

② $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ too.

- Also. If ① $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ has indeterminiant form $\frac{0}{0}$, or $\frac{\infty}{\infty}$

and ② $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$.

Then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ too.