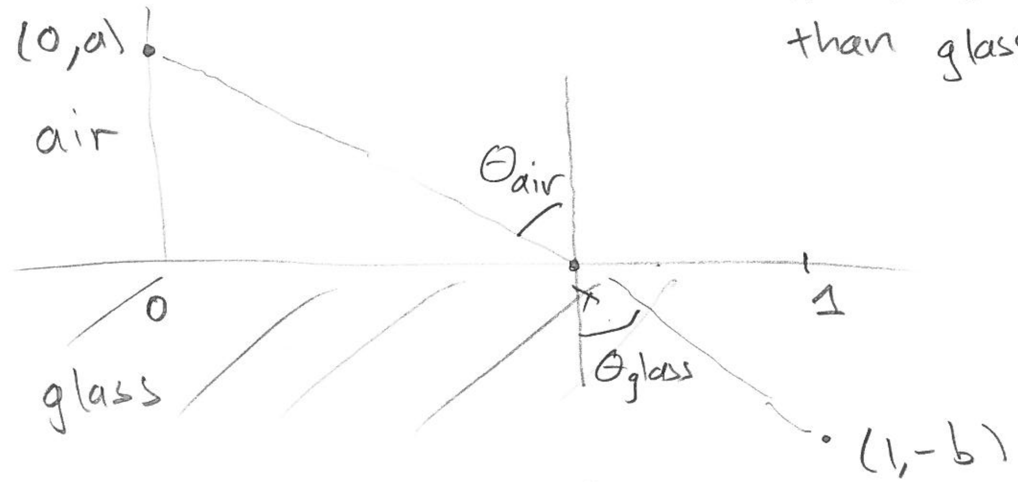
Week 10 Friday L04 11am

# Snell's Law

Light travels about 1.5 times faster in air than glass

$$\frac{V_{air}}{V_{glass}} = \frac{3}{2}.$$



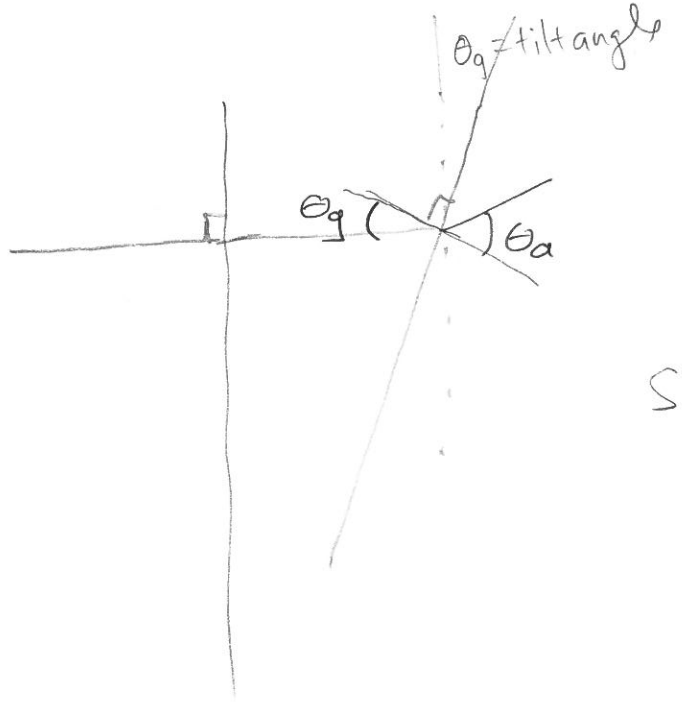

$$T(x) = \frac{\sqrt{a^2+x^2}}{V_{air}} + \frac{\sqrt{b^2+(1-x)^2}}{V_{glass}}$$

Light goes to point x which minimizes total travel time

$T'(x)=0$ ... The condition on point x is

$$\frac{3}{2} = \frac{V_{air}}{V_{glass}} = \frac{\sin(\theta_{air})}{\sin(\theta_{glass})}$$ Snell's Law.

Eyeglasses use this law.

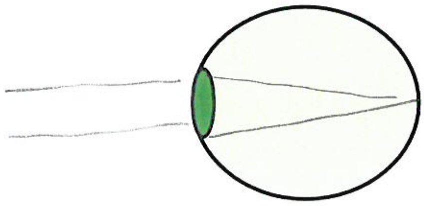



$$\frac{\sin(\theta_g)}{\sin(\theta_a)} = \frac{V_{glass}}{V_{air}} \quad \frac{2}{3}$$

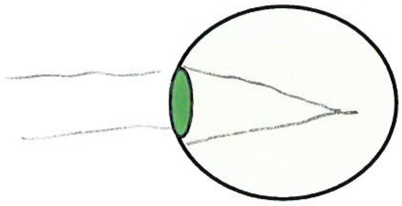
$$\sin(\theta_a) = \frac{3}{2}\sin(\theta_g)$$

$$\theta_{air} = \arcsin\left(\frac{3}{2}\sin(\theta_g)\right)$$

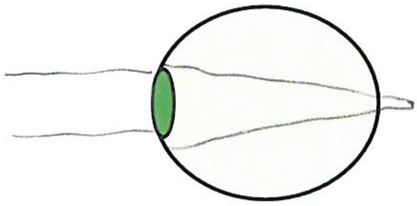
$\theta_g$ is amount we tilt back face of glass



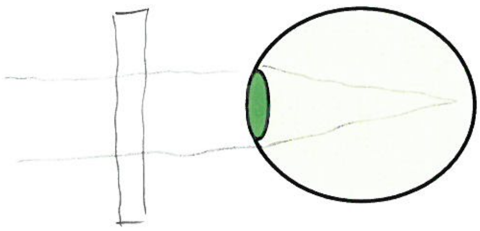
Normal



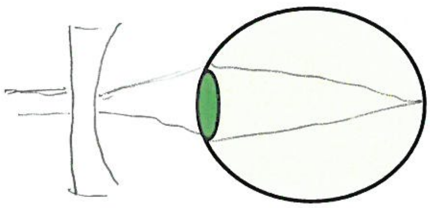
Nearsighted



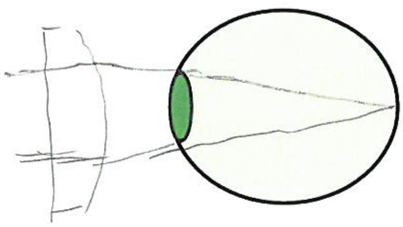
Farsighted



Normal



Nearsighted



Farsighted

# Limits and L'hopital rule

Example. Find $\lim_{x \to 1} \frac{\ln(x)}{x^2-1}$.

$\ln(x)$, $x^2-1$ both continuous.
Try to plug in $x=1$.
$\ln(1)=0$, $x^2-1\big|_{x=1} = 1^2-1=0$.

Get $\frac{0}{0}$ indeterminant.

To find limit we manipulate and connect it with derivaties

$$\frac{\ln(x)}{x^2-1} = \frac{\ln(x)-0}{(x^2-1)-0} = \frac{\ln(x)-\ln(1)}{(x^2-1)-(1^2-1)} = \frac{\frac{\ln(x)-\ln(1)}{x-1}}{\frac{(x^2-1)-(1^2-1)}{x-1}}$$

$\frac{\ln(x)-\ln(1)}{x-1} \xrightarrow{\text{as } x\to 1} 1$

$\frac{(x^2-1)-(1^2-1)}{x-1} \to 2$ as $x \to 1$.

As $x \to 1$, the quantity $\frac{\ln(x)-\ln(1)}{x-1} \longrightarrow$ tangent slope of $\ln(x)$ at input 1

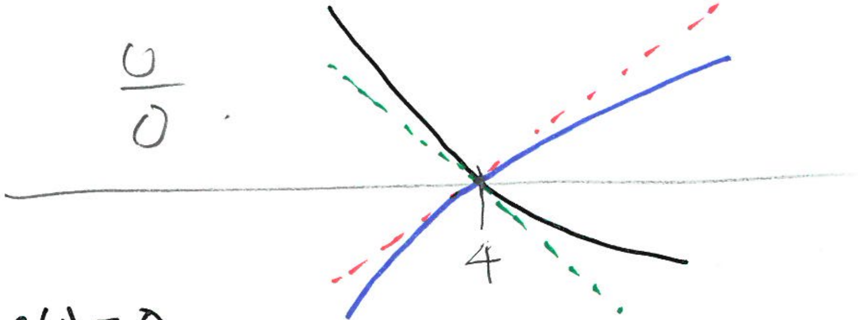
derivatie $\frac{1}{x} \longrightarrow \frac{1}{1}$ at input 1

Also as $x \to 1$, the quantity $\frac{(x^2-1)-(1^2-1)}{x-1} \longrightarrow$ tangent slope of $(x^2-1)$ at input

So $\lim_{x\to 1} \frac{\ln(x)}{x^2-1} = \lim_{x\to 1} \frac{\frac{\ln(x)-\ln(1)}{x-1}}{\frac{(x^2-1)-(1^2-1)}{x-1}} = \frac{1}{2}$ derivatie $2x-0$ so at input 1, we get 2.

# WW7 # 5

Find $\lim\limits_{x \to 4} \dfrac{f(x)}{g(x)}$ $\quad \dfrac{0}{0}$.




$\lim\limits_{x \to 4} f(x) = 0$, $\lim\limits_{x \to 4} g(x) = 0$

So we have indeterminant limit.

$$f(x) = f(x) - f(4)$$
$$g(x) = g(x) - g(4)$$

so $\dfrac{f(x)}{g(x)} = \dfrac{f(x) - f(4)}{g(x) - g(4)} = \dfrac{\dfrac{f(x)-f(4)}{(x-4)}}{\dfrac{g(x)-g(4)}{(x-4)}}$ $\quad \dfrac{f(x)-f(4)}{(x-4)} \to 1.2$, $\quad \dfrac{g(x)-g(4)}{(x-4)} \to -0.6$

As $x \to 4$, quantity $\dfrac{f(x)-f(4)}{x-4} \to$ tangent slope at $x=4$.
It is given as $1.2$

Also $x \to 4$, quantity $\dfrac{g(x)-g(4)}{x-4} \to$ tangent slope at $x=4$
Given as $-0.6$

Conclude $\lim\limits_{x \to 4} \dfrac{f(x)}{g(x)} = \dfrac{1.2}{-0.6} = -2$.

## L'hopital's Rule

Suppose $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ satisfying

① $\lim\limits_{x \to a} f(x) = 0$ AND $\lim\limits_{x \to a} g(x) = 0$ indeterminant limit ✓

② $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$

Then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = L$ too.

WW7 #6 Find $\lim\limits_{x \to 0} \frac{x^3}{\sin x - x}$

$f(x) = x^3 \to 0$ as $x \to 0$

$g(x) = \sin x - x \to 0$ as $x \to 0$

so limit is indeterminant

$f'(x) = 3x^2$, $g'(x) = \cos x - 1$

$\to 3 \cdot 0^2 = 0$ $\quad \to 1 - 1 = 0$ AGAIN indeterminant

$f''(x) = 3 \cdot 2 \cdot x$, $g''(x) = -\sin x$

$\to 3 \cdot 2 \cdot 0$ $\quad \to -0$

But $\frac{3 \cdot 2 x}{-\sin x}$ has lim $\frac{3 \cdot 2}{-1}(1)$ as $x \to 0$ $\qquad \frac{x}{\sin x} \to 1$

$\lim\limits_{x \to 0} \frac{f''(x)}{g''(x)} = \lim\limits_{x \to 0} \left(\frac{3 \cdot 2}{-1}\right) \frac{x}{\sin x} = \frac{-3 \cdot 2}{1}$

so $\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{-3 \cdot 2}{1}$ too, so $\lim\limits_{x \to 0} \frac{f(x)}{g(x)} = -6$ too.

#7 Find $\lim_{x \to 3} \dfrac{\ln(x/3)}{x^2-9}$

$f(x) = \ln\left(\frac{x}{3}\right)$

$g(x) = x^2-9$

$\lim_{x \to 3} f(x) = \ln\left(\frac{3}{3}\right) = \ln(1) = 0$, $\lim_{x \to 3} g(x) = 3^2 - 9 = 0$ $\quad \frac{0}{0}$ indeterminal

$f'(x) = \frac{1}{x}$, $g'(x) = 2x$

Then $\lim_{x \to 3} \dfrac{\frac{1}{x}}{2x} = \dfrac{\frac{1}{3}}{2 \cdot 3} = \dfrac{1}{2 \cdot 3^2} = \dfrac{1}{18}$

So $\lim_{x \to 3} \dfrac{f'(x)}{g'(x)} = \dfrac{1}{18}$. L'hopital rule says $\lim_{x \to 3} \dfrac{f(x)}{g(x)} = \dfrac{1}{18}$ too.

WARNING: In applying L'hopital's rule you <u>must</u> check condition that the original limit is indeterminant.

Example $\displaystyle\lim_{x \to \frac{\pi}{3}} \frac{\left(\frac{1}{2} - \cos(x)\right)}{\left(\sin(x)\right)} = \frac{0}{\left(\frac{\sqrt{3}}{2}\right)}$

$= 0$

correct

original limit not $\frac{0}{0}$.

$$\boxed{\begin{aligned} f(x) &= \tfrac{1}{2} - \cos(x) \\ f\left(\tfrac{\pi}{3}\right) &= \tfrac{1}{2} - \tfrac{1}{2} = 0 \\ g(x) &= \sin(x) \\ g\left(\tfrac{\pi}{3}\right) &= \tfrac{\sqrt{3}}{2}. \end{aligned}}$$

If we blindly apply L'hopital's rule.

$$\frac{0 + \sin(x)}{\cos(x)} = \tan(x), \text{ so } \lim_{x \to \frac{\pi}{3}} \tan(x) = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

Might then say $\cancel{\displaystyle\lim_{x \to \frac{\pi}{3}} \frac{\frac{1}{2} - \cos(x)}{\sin(x)} = \sqrt{3}}$ FALSE.

Other versions of L'hopital's rule.

$\lim_{x \to a} \frac{f(x)}{g(x)}$ has indeterminant form $\frac{\infty}{\infty}$

and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ too.

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Also if $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is indeterminant $\frac{\infty}{\infty}$ or $\frac{0}{0}$.

and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ too.

WW7 #10 $f$ continuous, $f(7)=0$

$f'$ has $f'(7)=14$.

Evaluate $\lim_{x\to 0} \dfrac{f(7+3x)+f(7+5x)}{x}$ $\to \dfrac{0}{0}$ indeterminant

As $x\to 0$ we have $f(7+3x)+f(7+5x) \longrightarrow f(7)+f(7)=0+0=0$.

and as $x\to 0$, we have $x\to 0$.

$$\frac{\text{derivative of top}}{\text{derivative of bottom}} = \frac{f'(7+3x)\cdot 3 + f'(7+5x)\, 5}{1}$$

$$\longrightarrow \frac{f'(7)\cdot 3 + f'(7)\cdot 5}{1} = \frac{14\cdot 3 + 14\cdot 5}{1} = 14\cdot 8.$$