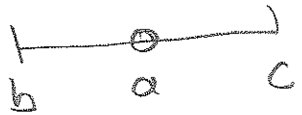
Week 11 Mon L03 2pm

Midterm exam viewing. Wednesday (14 Nov) 4-6pm
Math Support Center (3011-3013 lift 2 or 3).

L'hopital's rule. Suppose $b \le x \le c$ is closed interval



Suppose f, g differentiable on $[b,a) \cup (a,c]$

and ① $\lim_{x \to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (indeterminant)

② $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ too.

Mean Value Theorem

$\boxed{\frac{f(b)-f(a)}{b-a}}$ = secant slope from $(a, f(a))$ to $(b, f(b))$

There is interior point c so

$\frac{f(b)-f(a)}{b-a} = f'(c)$.

Reason/proof of L'hopital's rule.

Cauchy (2 function) Mean Value Theorem

Have 2 functions f, g differentiable

$\frac{f(b)-f(a)}{g(b)-g(b)}$ As we move from one endpoint a to other b, there is interior d so that $\frac{f'(d)}{g'(d)} =$

Proof of Cauchy Mean Value Theorem.

$$h(x) = f(x)\,(g(b)-g(a)) \quad - (f(b)-f(a))\,g(x)$$

These constants $(g(b)-g(a))$ and $(f(b)-f(a))$ are trickily choosen so that

$$h(a) = h(b) \;\left(= f(b)g(a) - f(a)g(b)\right)$$ SAME VALUE at endpoints.

Function h is differentiable ($\Rightarrow$ continuous).

EVT, LEVT, there is interior point d so it is local max or min

and so $\underline{\underline{h'(d) = 0}}$.

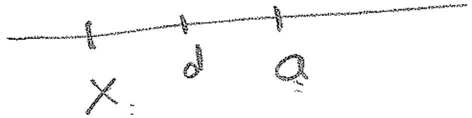
So $h'(d) = 0 = f'(d)\,(g(b)-g(a)) - (f(b)-f(a))\,g'(d)$

so $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(d)}{g'(d)}$$

L'hopital rule reason/proof.

Case $\frac{0}{0}$. So $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$.

Define $f(a) = 0$, $g(a) = 0$.



$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} \overset{\text{CMVT}}{=} \frac{f'(d)}{g'(d)}$$

As $x \to a$, since d is trapped between x and a we have $d \to a$

But $\lim_{d\to a} \frac{f'(d)}{g'(d)} = L$, so we conclude $\lim_{x\to a} \frac{f(x)}{g(x)} = L$. too.

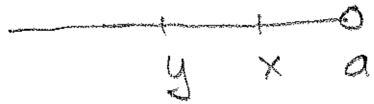
Case $\frac{\infty}{\infty}$. See notes.

The $\frac{\infty}{\infty}$ case of L'hopital's rule

Unlike the $\frac{0}{0}$ case we can <u>not</u> set $f(a)=\infty$, $g(a)=\infty$.

We show $\lim\limits_{x\to a^-} \frac{f(x)}{g(x)} = L$.

We take $y < x < a$



and apply the Cauchy Mean Value Theorem to $y<x$. There is an interior point d $(y < d < x)$ so that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(d)}{g'(d)}.$$

Write $\frac{f(x)-f(y)}{g(x)-g(y)}$ as $\frac{f(x)\cdot\left(1-\frac{f(y)}{f(x)}\right)}{g(x)\cdot\left(1-\frac{g(y)}{g(x)}\right)}$ $\left(= \frac{f'(d)}{g'(d)}\right)$.

So $$\frac{f(x)}{g(x)} = \frac{f'(d)}{g'(d)} \cdot \frac{\left(1-\frac{g(y)}{g(x)}\right)}{\left(1-\frac{f(y)}{f(x)}\right)}$$

Now as $y \to a$, we have for any $y < d < a$ that $d \to a$, and so

$$\lim_{d \to a} \frac{f'(d)}{g'(d)} = L.$$

Fix y so that $\frac{f'(d)}{g'(d)}$ is near to L for $y < d < a$.

Now let $x \to a$, and note that $y < d < x < a$ in

$$\frac{f(x)}{g(x)} = \frac{f'(d)}{g'(d)} \frac{\left(1 - \frac{g(y)}{g(x)}\right)}{\left(1 - \frac{f(y)}{f(x)}\right)}$$

As $x \to a^-$, the terms $\left(1 - \frac{g(y)}{g(x)}\right) \to 1 - 0 = 1$, and $\left(1 - \frac{f(y)}{f(x)}\right) \to 1 - 0 = 1$.

We conclude $\frac{f(x)}{g(x)}$ is near to L. If we let BOTH $y \to a$, and $y < x \to a$,

we can deduce $\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{d \to a^-} \frac{f'(d)}{g'(d)} = L$.

Student problem Find $\lim_{x \to \infty} \left(x - x^2 \ln\left(1+\frac{1}{x}\right) \right)$

$\infty - \infty^2 \cdot 0$ indeterminant.

Do change of variables $x = \frac{1}{h}$ $\left(h = \frac{1}{x}\right)$

As $x \to \infty$, $h \to 0$.

$$\lim_{h \to 0} \left(\frac{1}{h} - \left(\frac{1}{h}\right)^2 \ln(1+h) \right) = \lim_{h \to 0} \frac{h - \ln(1+h)}{h^2} \qquad \frac{0 - \ln(1+0)}{0^2} = \frac{0-0}{0}$$

$$\frac{h}{h^2} - \frac{\ln(1+h)}{h^2}$$

Can try L'hopital rule

1st try of L'hopital's rule: $\dfrac{1 - \frac{1}{1+h}(0+1)}{2h} = \dfrac{1 - \frac{1}{1+h}}{2h}$ $\begin{matrix} \to 1 - \frac{1}{1+0} = 1-1 = 0 \\ \to 2 \cdot 0 = 0 \end{matrix}$ $\frac{0}{0}$

Again indeterminant

2nd try with L'hopital's rule

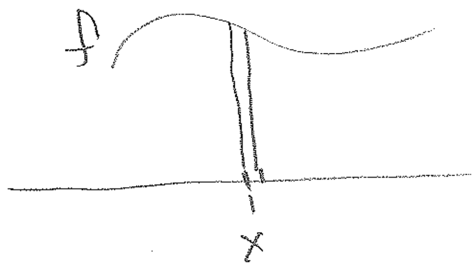
$$\frac{0 - (-1)(1+h)^{-2}}{2} \longrightarrow \frac{0 + \frac{1}{(1+0)^2}}{2} = \frac{1}{2}.$$

So $\lim_{x \to \infty} \left(x - x^2 \ln\left(1+\frac{1}{x}\right) \right) = \frac{1}{2}$.

Antiderivatives

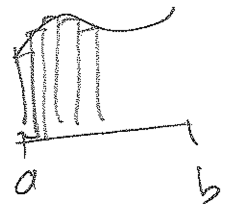
Differential calculus. Differentials dy, dx are "infinitesimals"
Ratio $\frac{dy}{dx}$ has meaning. Tangent slope.

Integral calculus. Infinitesimal dx Think of dx as base of infinitesimal rectangle. For height take $f(x)$.



Area = height · base = $f(x)dx$

Now add up ∞ many such areas



$$\int_a^b f(x)dx = \text{area (definite integral)}.$$

The most important theorem of calculus (Fundamental Theorem of Calculus) will say. derivatives and antiderivatives give connection between differential and integral calculus.

Given function f on an interval, and <u>antiderivative</u> is a function F so that $F'(x) = f(x)$.

Examples ① $f(x) = x^2$ Need $F'(x) = x^2$

$F(x) = \frac{x^3}{3}$ has $F'(x) = \frac{3 \cdot x^2}{3} = x^2$.

② $f(x) = 0$ (zero function)

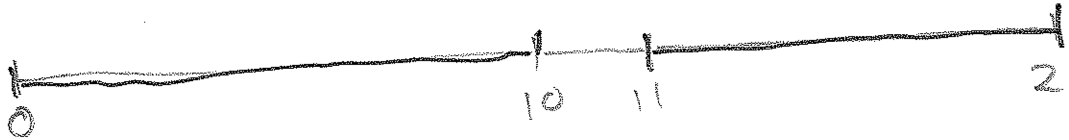
Antiderivative. $F(x) =$ <u>any</u> <u>constant</u> has derivative 0.

Family of antiderivatives.

③ On the interval $[0, 10]$ x^2 has $\frac{x^3}{3} + C$ as antiderivative.

Recall from MVT. If $F(x)$ and $G(x)$ have same derivative on interval $[a, b]$, then $G(x) - F(x)$ is constant.

3' Consider domain $[0,10] \cup [11,21]$



$f(x) = x^2$

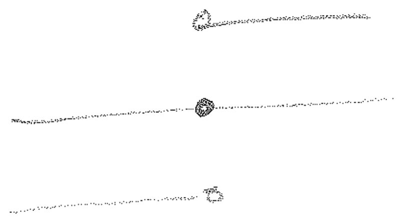
$$F(x) = \begin{cases} \dfrac{x^3}{3} + 9 & \text{on } 0 \le x \le 10 \\ \dfrac{x^3}{3} & \text{on } 11 \le x \le 21 \end{cases}$$

$$G(x) = \left\{ \dfrac{x^3}{3} + 1 \quad \text{on } 0 \le x \le 10 \ \underline{\text{and}} \ 11 \le x \le 21 \right.$$

F, G are both antiderivatives of x^2. But $F - G$ is not constant on ALL of $[0,10] \cup [11,21]$

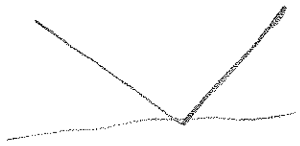
<u>Remark</u> Not every function has antiderivative.

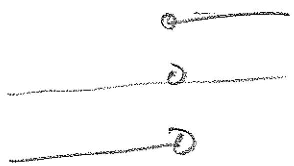
$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$



f does NOT have antiderivative

It is <u>NOT</u> $F(x) = |x|$





If f is <u>continuous</u> then f has antiderivative.

WW8 #1 Notation for antiderivative.

If g is a function which has an antiderivative, we use "strange notation"

$$\int g(x)\,dx$$

for the family of antiderivatives.

Find derivative of $f(x) = -\frac{\sqrt{x^2+36}}{36x} + C$.

If $g(x) = f'(x)$, then

$$\int g(x)\,dx = -\frac{\sqrt{x^2+36}}{36\,x} + C.$$

We have $f'(x) = \left(-\frac{1}{36}\right)\left(\left(\sqrt{x^2+36}\right)' \cdot x^{-1} + \left(\sqrt{x^2+36}\right) \cdot \left(x^{-1}\right)'\right) + 0$

$$= \left(-\frac{1}{36}\right)\left(\frac{1}{\cancel{2}\sqrt{x^2+36}}\,\cancel{2}\cancel{x}\left(\frac{1}{\cancel{x}}\right) + \left(\sqrt{x^2+36}\right)(-1)\,x^{-2}\right).$$

$$= \left(-\frac{1}{36}\right)\left(\frac{1}{\sqrt{x^2+36}} - \frac{\sqrt{x^2+36}}{x^2}\right)$$

#2 Find

(a) $\int (14t - 5t^2 + 9)\, dt = 14\frac{t^2}{2} - 5\frac{t^3}{3} + 9t + C$

$= 7t^2 - \frac{5}{3}t^3 + 9t + C$

(b) $\int (u^{-7/4} + 3u^{1/2})\, du = \frac{u^{-3/4}}{(-3/4)} + 3\frac{u^{3/2}}{(3/2)} + C$

(c) $\int \frac{1}{5}x^{-4}\, dx = \frac{1}{5} \cdot \frac{x^{-3}}{-3} + C$