Tentatively. Midterm exam viewing Wed (14 Nov) 4–6pm in Math Support Center. (3011–3013 lift 3 or lift 2)

L'hopital's rule Suppose f, g differentiable on $[b,a) \cup (a,c]$

and ① $\lim_{x \to a} \frac{f(x)}{g(x)}$ is indeterminant $\frac{0}{0}$, or $\frac{\infty}{\infty}$

② $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$ too.

Reason/proof why L'hopital's rule is true

Cauchy (or 2 function Mean Value Theorem)

Have 2 differentiable functions f, g (and assume $g' \neq 0$).

Look at $\frac{f(b) - f(a)}{g(b) - g(a)}$ (instead of secant slope $\frac{f(b) - f(a)}{b - a}$ from $(a, f(a)), (b, f(b))$

Then as we move from one endpoint to other, there will be an interior point c so that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Reason why CMVT is true. Instead of looking at difference of f and secant line (which we did for MVT), we look at

$$h(x) = (f(b)-f(a))\,g(x) - f(x)\,(g(b)-g(a))$$

Can check $h(a) = h(b) = f(b)g(a) - f(a)g(b)$

The function h is differentiable and $h(a)=h(b)$ so EVT ⇒ there is max/min, and LEVT ⇒ there is max/min in interior c, so $h'(c)=0$

Since $h'(x) = (f(b)-f(a))\,g'(x) - f'(x)\,(g(b)-g(a))$

$0 = h'(c) = \ldots$

so

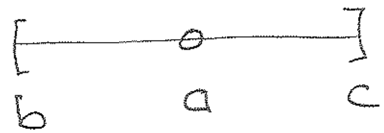
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

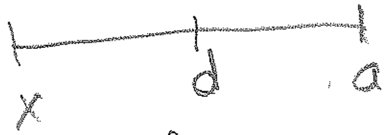
Use of <u>CMVT</u> to see why L'hopital's rule is true.

We will assume $\frac{0}{0}$ (so $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$)

Define $f(a) = 0$, $g(a) = 0$

f, and g are continuous.





$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} \overset{\text{CMVT}}{=} \frac{f'(d)}{g'(d)}$$ for a d between x and a

If we let $x \to a$, then $d \to a$. But as $d \to a$ $\lim_{d \to a} \frac{f'(d)}{g'(d)} = L$.

So $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

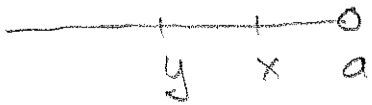
The case $\frac{\infty}{\infty}$ is similar but longer.

The $\frac{\infty}{\infty}$ case of L'hopital's rule

Unlike the $\frac{0}{0}$ case we can <u>not</u> set $f(a)=\infty$, $g(a)=\infty$.

We show $\lim\limits_{x\to a^-} \frac{f(x)}{g(x)} = L$.

We take $y < x < a$



and apply the Cauchy Mean Value Theorem to $y<x$. There is an interior point d $(y<d<x)$ so that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(d)}{g'(d)}.$$

Write $\frac{f(x)-f(y)}{g(x)-g(y)}$ as $\frac{f(x)\cdot\left(1-\frac{f(y)}{f(x)}\right)}{g(x)\cdot\left(1-\frac{g(y)}{g(x)}\right)}$ $\left(= \frac{f'(d)}{g'(d)}\right)$.

So

$$\frac{f(x)}{g(x)} = \frac{f'(d)}{g'(d)} \cdot \frac{\left(1-\frac{g(y)}{g(x)}\right)}{\left(1-\frac{f(y)}{f(x)}\right)}$$

Now as $y \to a$, we have for any $y < d < a$ that $d \to a$, and so

$$\lim_{d \to a} \frac{f'(d)}{g'(d)} = L.$$

Fix y so that $\frac{f'(d)}{g'(d)}$ is near to L for $y < d < a$.

Now let $x \to a$, and note that $y < d < x < a$ in

$$\frac{f(x)}{g(x)} = \frac{f'(d)}{g'(d)} \frac{\left(1 - \frac{g(y)}{g(x)}\right)}{\left(1 - \frac{f(y)}{f(x)}\right)}.$$

As $x \to a^-$, the terms $\left(1 - \frac{g(y)}{g(x)}\right) \to 1 - 0 = 1$, and $\left(1 - \frac{f(y)}{f(x)}\right) \to 1 - 0 = 1$.

We conclude $\frac{f(x)}{g(x)}$ is near to L. If we let BOTH $y \to a$, and $y < x \to a$,

we can deduce $\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{d \to a^-} \frac{f'(d)}{g'(d)} = L$

Student problem Find $\lim_{x\to\infty}\left(x - x^2\ln\left(1+\frac{1}{x}\right)\right)$

$\infty - \infty^2 \cdot 0$ indeterminant

"change of variable" $x = \frac{1}{h}$. So $x \to \infty$, then $h \to 0$.

$$\lim_{h\to 0}\left(\frac{1}{h} - \left(\frac{1}{h}\right)^2 \ln(1+h)\right) = \lim_{h\to 0}\frac{h - \ln(1+h)}{h^2} \quad \begin{matrix}\to 0-0\\ \to 0^2\end{matrix}$$

Indeterminant $\frac{0}{0}$. Try to apply L'hopital's rule

$$\frac{1 - \frac{1}{1+h}\cdot(0+1)}{2h} = \frac{1 - \frac{1}{1+h}}{2h} \text{ as } h \to 0 \text{ we get } \frac{1 - \frac{1}{1+0}}{2\cdot 0} \begin{matrix}= 0\\ = 0\end{matrix}$$

Indeterminant $\frac{0}{0}$. Try 2nd time

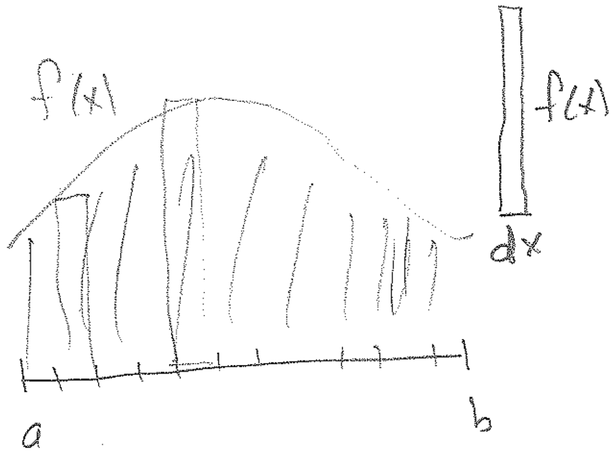
$$\frac{0 - (-1)(1+h)^{-2}}{2} \text{ as } h \to 0 \text{ we get } \frac{\frac{1}{(1+0)^2}}{2} = \frac{1}{2}$$

So $\lim_{x\to\infty}\left(x - x^2\ln\left(1+\frac{1}{x}\right)\right) = \frac{1}{2}$.

Antiderivatives.

Differential calculus. $\frac{dy}{dx}$ dy: "infinitesimal" dx: "infinitesimal". $\frac{dy}{dx}$ ratio makes sense.

Integral calculus dx "infinitesimal". think of as base of rectangle
$f(x)$ think as height of rectangle



area $= f(x)dx$ ("very small")

Add up "lots" of "infinitesimal" ~~re~~rectangle area. $\int_a^b f(x)dx$ = area.

There is extremely important connection between differential and integral calculus (Fundamental Theorem of Calculus). The two are related by antiderivatives.

Suppose we have a function f. An antiderivative of f is <u>any</u> function F so that $F' = f$.

Examples ① $f(x) = x^2$ on $(-\infty, \infty)$

$F(x) = \frac{x^3}{3}$ is an antiderivative since $F'(x) = \left(\frac{x^3}{3}\right)' = \frac{3 \cdot x^2}{3}$
$= f(x)$.

② $f(x) = 0$ on $(-\infty, \infty)$

$F(x) = 1$, or 2, or in fact any constant C.

③ $\frac{x^3}{3} + C$ is also an antiderivative of x^2.

If F is an antiderivative of f (so $F' = f$), then adding a <u>constant</u> C to F gives another antiderivative $F + C$.

By MVT, if f is function on $a \le x \le b$ and F_1, F_2 are two antiderivatives, then they differ by constant.

WW 8 #1. Find derivative of $f(x) = -\frac{\sqrt{x^2+36}}{36x} + C$.

Take derivative prod (of $\sqrt{x^2+36}$ and x^{-1}).

$$f'(x) = \left(-\frac{1}{36}\right)\left(\cancel{\tfrac{1}{2}}(x^2+36)^{-1/2}(\cancel{2x}) \cdot \frac{1}{\cancel{x}} + \sqrt{x^2+36}\,(-1)(x^{-2})\right) + 0.$$

$$f'(x) = \left(-\frac{1}{36}\right)\left(\frac{1}{\sqrt{x^2+36}} - \frac{\sqrt{x^2+36}}{x^2}\right)$$

So antiderivative of $\boxed{\left(-\frac{1}{36}\right)\left(\frac{1}{\sqrt{x^2+36}} - \frac{\sqrt{x^2+36}}{x^2}\right)}$ is $-\frac{\sqrt{x^2+36}}{36x} + C$.

We use notation $\int g(x)\,dx$ to denote an antiderivative of g.

$$\int \boxed{\left(-\frac{1}{36}\right)\left(\frac{1}{\sqrt{x^2+36}} - \frac{\sqrt{x^2+36}}{x^2}\right)}\,dx = -\frac{\sqrt{x^2+36}}{36x} + C$$

Similarly $\int x^2\,dx = \frac{x^3}{3} + C$

$$\int 0\,dx = C.$$

#2 Find

(a) $\int (14t - 5t^2 + 9)\, dt = 7t^2 - \frac{5t^3}{3} + 9t + C.$

(b) $\int \left(\frac{1}{u^{7/4}} + 3u^{1/2}\right) du = \int (u^{-7/4} + u^{1/2})\, du$

$-\frac{3}{4}$

$$= \frac{u^{(-7/4+1)}}{(-3/4)} + \frac{3u^{3/2}}{3/2} + C.$$

(c) $\int \frac{1}{5x^4}\, dx = \int \frac{1}{5} x^{-4}\, dx = \frac{1}{5}\frac{x^{-3}}{-3} + C$

$$= \frac{x^{-3}}{-15} + C.$$