Suppose f is function on domain $a \le x \le b$

n integer

$\Delta x = \dfrac{b-a}{n}$

Divide $a \le x \le b$ into n equal length subintervals.

Pick points $x_1, x_2, \dots, x_n$ for the n subintervals

The sum $\Delta x\, f(x_1) + \Delta x\, f(x_2) + \dots + \Delta x\, f(x_n) = \sum_{i=1}^{n} \Delta x \cdot f(x_i)$

is called Riemann Sum. Approximation for area.

Notation: sigma notation.

If x_i are Left endpoint call RS the Left RS.

x_i — Right —— Right RS

x_i — mid point —— Midpoint RS.

If f is continuous, we can take x_i^{max} to be the point which yield max on ith subinterval. Similarly for min x_i^{min}

$$\underbrace{\sum_{i=1}^{n} \Delta x_i f(x_i^{min})}_{\text{underestimate}} \leq \overset{\text{Any other}}{\sum_{i=1}^{n} \Delta x f(x_i)} \leq \underbrace{\sum_{i=1}^{n} \Delta x f(x_i^{max})}_{\text{overestimate}}$$

If as $n \to \infty$ the gap between two goes to zero, we say function is integrable, the $\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) = L$ is called the definite integral. We use notation

$$\int_a^b f(x)\,dx$$

the value. Intuition is this is area.

Example $f(x)=x^2$ $\quad 0 \le x \le b$

$n \rightsquigarrow$ gives $\Delta x = \frac{b-0}{n} = \frac{b}{n}$

Left Riemann Sum is

$$\Delta x \left(0^2 + \left(\frac{b}{n}\right)^2 + \left(2\frac{b}{n}\right)^2 + \dots + \left((n-1)\frac{b}{n}\right)^2 \right)$$

$$= \frac{b}{n}\left(0^2 + \left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)^2 2^2 + \left(\frac{b}{n}\right)^2 3^2 + \dots + \left(\frac{b}{n}\right)^2 (n-1)^2 \right)$$

$$= \left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 \sum_{i=0}^{n-1} i^2$$

Right RS is $\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 \sum_{i=1}^{n} i^2$

What happens as $n \to \infty$?

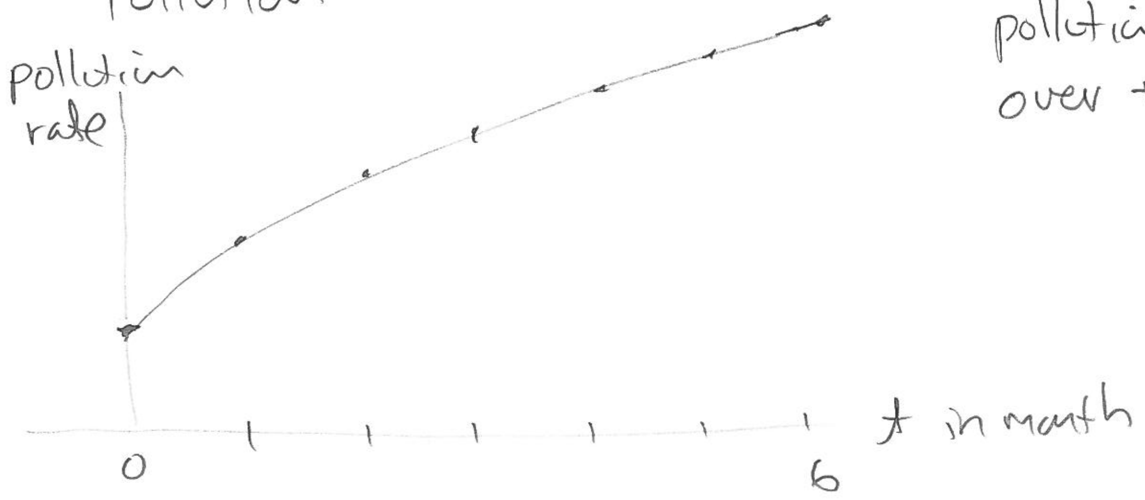
$$\left(1^2+2^2+\dots+n^2\right) = \frac{n(n+1)(2n+1)}{6}, \quad \left(1^2+2^2+\dots+(n-1)^2\right) = \frac{(n-1)n(2n-1)}{6}$$

$$LRB = \left(\frac{b}{n}\right)^3 \frac{(n-1)n(2n-1)}{6} = \frac{b^3}{3}\left(1-\frac{1}{n}\right) 1 \frac{\left(2-\frac{1}{n}\right)}{2} \to \frac{b^3}{3} \text{ as } n\to\infty$$

$$RRS = \left(\frac{b}{n}\right)^3 \frac{(n)(n+1)(2n+1)}{6} = \frac{b^3}{3}(1)\left(1+\frac{1}{n}\right)\frac{\left(2+\frac{1}{n}\right)}{2} \to \frac{b^3}{3} \text{ as } n\to\infty$$

So $\displaystyle\int_0^b x^2\,dx = \frac{b^3}{3}$.

WW 8 # 6 Pollution



pollution rate increases over time.

Since function increasing LRS underestimate, RRS over estimate } difference is $\left(f(6)-f(0)\right)\cdot\Delta x$

If we want to know pollution for 6 months to accuracy of 1 ton, How often should we same pollution rate?

Need $\left(f(6)-f(0)\right)\cdot\Delta x \le 1$ (ton).

$(51 - 8)\cdot\frac{6}{n} \le 1$ so $43\cdot 6$ times for 6 months.

43 times per month.

Cases where the definite integral exists

Examples ① When function f increasing on $a \le x \le b$.

$$\text{Left RS} \le \text{Any RS} \le \text{Right RS}$$

Gap between two is $(f(b) - f(a))\,\Delta x$ $\qquad \Delta x = \frac{b-a}{n}$.

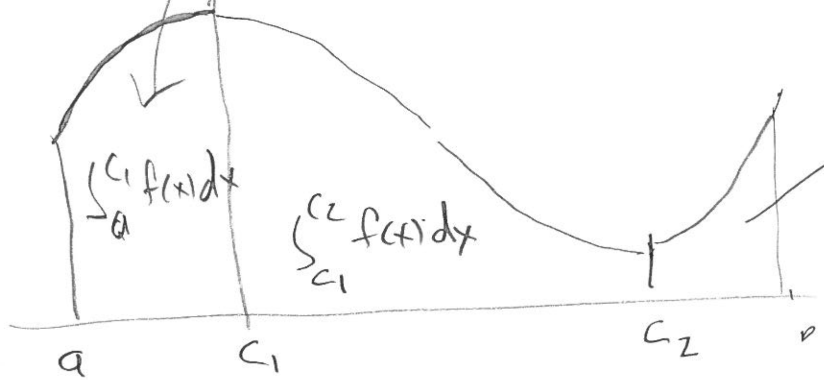
$$= (f(b) - f(a))\left(\frac{b-a}{n}\right) \longrightarrow 0 \text{ as } n \to \infty.$$

Conclude Riemann Sums have a limit.

② Same for decreasing.

③ Combine ① and ②

f increasing on this interval, so definite integral exist.

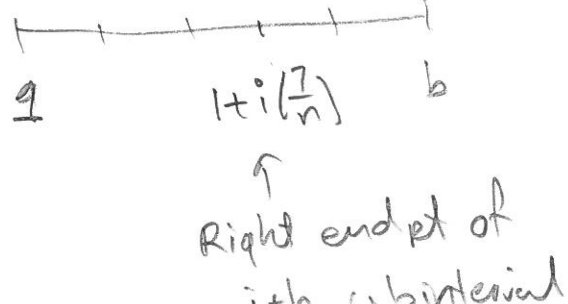


$\int_{c_2}^{b} f(x)dx$ exists

WW8 # 9 For definite integral $\int_1^b f(x)\,dx$, a RS is

$$\text{Right RS} = \frac{1}{(1+\frac{7}{n})}\left(\frac{7}{n}\right) + \frac{1}{(1+2(\frac{7}{n}))}\cdot\left(\frac{7}{n}\right) + \dots + \frac{1}{(1+n(\frac{7}{n}))}\cdot\left(\frac{7}{n}\right)$$

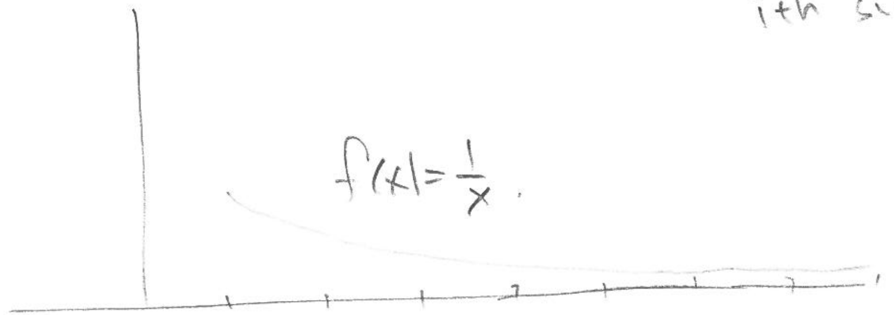
$\Delta x = \frac{7}{n}$. But $\Delta x = \frac{b-1}{n}$. so $b = 8$



$$\frac{1}{1+i(\frac{7}{n})} = f\left(1+i\left(\frac{7}{n}\right)\right) \quad \text{so } f(x) = \frac{1}{x}$$

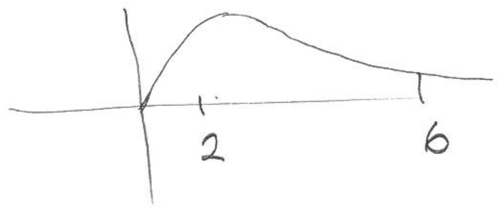
$$\int_1^8 \frac{1}{x}\,dx$$

$$\Delta x = \left(\frac{8-1}{n}\right)$$



#10 Represent definite integral $\int_2^6 \frac{x}{1+x^5}\,dx$

as a limit of Riemann Sums.



Since f increases followed by decrease
it is integrable. $\Delta x = \frac{6-2}{n} = \frac{4}{n}$.

Take Right Riemann Sums. $2 + i\Delta x = 2 + i\left(\frac{4}{n}\right)$ is right endpoint of ith subinterval

$$f\left(2+i\left(\tfrac{4}{n}\right)\right) = \frac{2+i\left(\frac{4}{n}\right)}{1+\left(2+i\left(\frac{4}{n}\right)\right)^5}$$

$$\sum_{i=1}^{n} \frac{2+i\left(\frac{4}{n}\right)}{1+\left(2+i\left(\frac{4}{n}\right)\right)^5} \cdot \left(\frac{4}{n}\right)$$

This matches with multiple choice answer F.

Cases where f is integrable

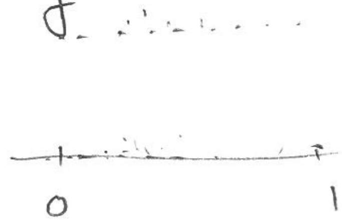
1. f increasing or decreasing
2. <u>finite</u> combo of increasing / decreasing
3. f continuous.

} covers most functions.

<u>Example</u> where f is NOT integrable. $0 \le x \le 1$

$$f(x) = \begin{cases} 0 & \text{when irrational} \\ 1 & \text{when rational} \end{cases}$$

very discontinuous



Riemann sums for f may not have limit.