WW7 #9 Find $\lim_{t \to +\infty} \left(\frac{3^t + 6^t}{3}\right)^{1/t}$.

In $\left(\frac{3^t+6^t}{3}\right)^{1/t}$ bring out 6^t:

$$= \left(\frac{6^t\left(\left(\frac{3}{6}\right)^t+1\right)}{3}\right)^{1/t} = \left(6^t\right)^{1/t}\left(\frac{\left(\frac{3}{6}\right)^t+1}{3}\right)^{1/t} = 6\left(\frac{\left(\frac{1}{2}\right)^t+1}{3}\right)^{1/t}.$$

As $t \to +\infty$, the term $\left(\frac{1}{2}\right)^t \to 0$. So $\frac{\left(\frac{1}{2}\right)^t+1}{3} \to \frac{0+1}{3} = \frac{1}{3}$

Since $\frac{1}{t} \to 0$ and $\frac{\left(\frac{1}{2}\right)^t+1}{3} \to \frac{1}{3}$, we have

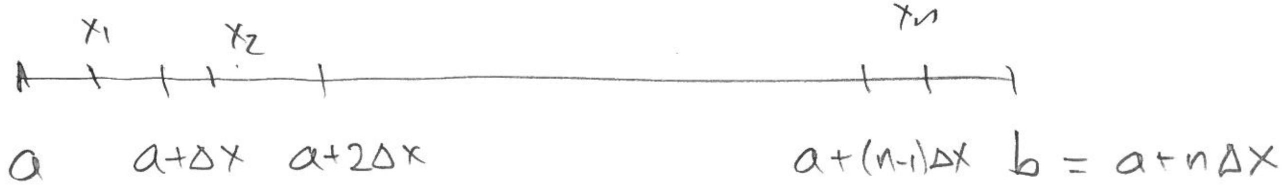
$$\left(\frac{\left(\frac{1}{2}\right)^t+1}{3}\right)^{\frac{1}{t}} \to \left(\frac{1}{3}\right)^0 = 1$$

Conclude $\lim_{t \to +\infty} \left(\frac{3^t+6^t}{3}\right)^{1/t} = 6 \cdot 1 = 6$.

Riemann Sums

f a function with domain $a \le x \le b$.

Take integer n,



divide interval into n subintervals of length $\Delta x = \dfrac{b-a}{n}$.

Pick points $x_1, x_2, \dots, x_n$ from the subintervals.

The sum $\Delta x \cdot f(x_1) + \Delta x f(x_2) + \dots + \Delta x f(x_n)$

is called a Riemann Sum. It is approximation for area.

If we choose x_i to be RIGHT endpoint, then RS is called Right Riemann Sum.

We can also choose left endpoint, midpoint, etc

We use <u>summation notation</u> $\displaystyle\sum_{i=1}^{n} (\Delta x) f(x_i)$ shorthand for

If function is continuous. We can pick x_i^{max} to be point the ith interval which gives function max, then "obviously" the RS

$$\sum_{i=1}^{n} \Delta x \, f(x_i^{max}) \qquad \text{"upper estimate"}$$

is larger than any other RS (with n sibintervals)

We can also pick x_i^{min} for min

$$\sum_{i=1}^{n} \Delta x \, f(x_i^{min}) \qquad \text{"lower estimate"}.$$

We say a function f is <u>integralele</u> over interval $a \le x \le b$, if as $n \to \infty$, <u>any</u> Riemann sum $\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \, f(x_i)$ has limit.

<u>Example</u> $f(x) = x^2$ interval $0 \le x \le b$.

For n, we get $\Delta x = \frac{b-0}{n} = \frac{b}{n}$

Since f is increasing left RS is underestimate -
right RS is overestimate
and any other RS with n subintervals is between the two.

$$\text{Left RS} = \Delta x\, f(0) + \Delta x\, f\left(\frac{b}{n}\right) + \Delta x\, f\left(2\left(\frac{b}{n}\right)\right) + \cdots + \Delta x\, f\left((n-1)\left(\frac{b}{n}\right)\right)$$

$$= \Delta x \sum_{i=0}^{(n-1)} f\left(i\left(\frac{b}{n}\right)\right) = \Delta x \sum_{i=0}^{(n-1)} i^2 \left(\frac{b}{n}\right)^2 \qquad f\left(i\left(\frac{b}{n}\right)\right) = i^2\left(\frac{b}{n}\right)^2$$

$$= \left(\frac{b}{n}\right)^3 \cdot \left(\sum_{i=0}^{(n-1)} i^3\right) = \left(\frac{b}{n}\right)^3 \cdot \{0^2 + 1^2 + 2^2 + \cdots + (n-1)^2\}.$$

Similarly $\text{Right RS} = \Delta x \sum_{i=1}^{n} f\left(i\left(\frac{b}{n}\right)\right) = \left(\frac{b}{n}\right)^3 \{1^2 + 2^2 + \cdots + (n-1)^2 + n^2\}.$

Fact $\quad 1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$1^2+2^2+\cdots+(n-1)^2 = \frac{(n-1)(n)(2n-1)}{6}.$$

So Left RS is $\left(\frac{b}{n}\right)^3 \cdot \dfrac{(n-1)\cdot n\cdot(2n-1)}{6} = \dfrac{b^3}{3} \cdot \dfrac{(1-\frac{1}{n})\cdot 1\cdot(2-\frac{1}{n})}{2}$

Right RS is $\left(\frac{b}{n}\right)^3 \cdot \dfrac{n(n+1)(2n+1)}{6} = \dfrac{b^3}{3} \cdot \dfrac{1\cdot(1+\frac{1}{n})(2+\frac{1}{n})}{2}$

As $n \to \infty$, the terms $\dfrac{(1-\frac{1}{n})\cdot 1\cdot(2-\frac{1}{n})}{2} \to 1$

as does $\dfrac{1\cdot(1+\frac{1}{n})(2+\frac{1}{n})}{2} \to 1$

Since other RS are trapped we see $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\frac{b}{n}\right) f(x_i) \longrightarrow \dfrac{b^3}{3}$.

So $f(x) = x^2$ on $0 \le x \le b$, is integrable and its definite integral is $\dfrac{b^3}{3}$. We use notation $\displaystyle\int_a^b x^2\,dx$

RS $\sum_{i=1}^{n} \Delta x\, f(x_i) = \sum_{i=1}^{n} x_i^2 \Delta x$

As $n \to \infty$, we change Δx to dx.

x_i^2 — x^2

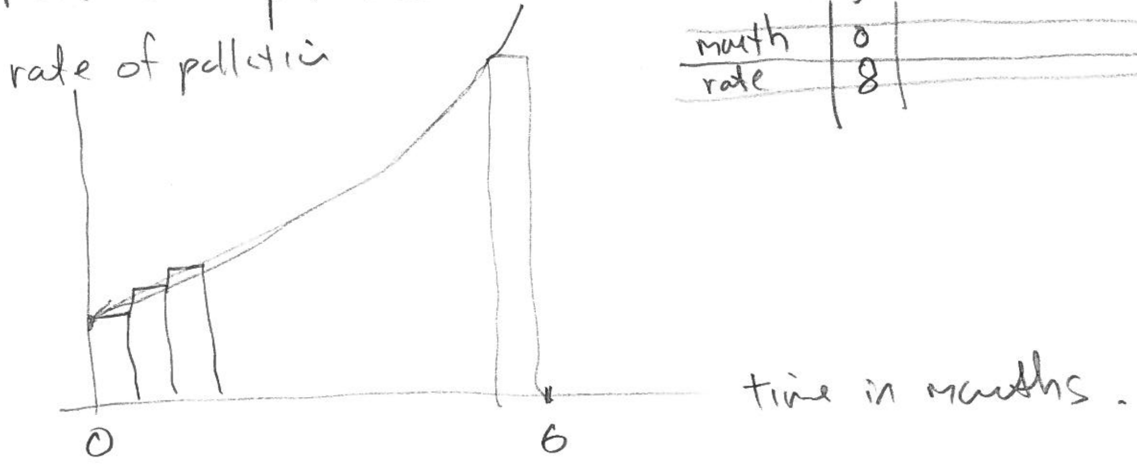
$\sum_{i=1}^{n}$ — $\int_a^b$

$$\sum_{i=1}^{n} x_i^2 \Delta x \longrightarrow \underline{\underline{\int_a^b x^2\, dx}}$$

Originally areas and definite integral were "completed" in this fashion. (letting $n \to \infty$).

Fundamental Theorem of Calculus, shows how to compute definite integrals using antiderivatives

WW8 #6 Pollution problem



month	0		6
rate	8		51

If we divide 6 into n equal subintervals (length $\frac{6-0}{n}$).

Left Riemann Sum is underestimate

Right ______ overestimate

$\Delta x = \frac{6}{n}$

Because f is <u>increasing function</u>

$$\text{Right RS} - \text{Left RS} = (f(6) - f(0)) \cdot \Delta x$$

$$(51 - 8)\,\Delta x$$

If we wand Right RS − Left RS $\leq$ 1 ton, we make

$(51-8)\,\Delta x \leq 1$, $\quad 43 \cdot \frac{6}{n} \leq 1 \quad$ so $\; 43 \cdot 6 \leq n$

so $43 \cdot 6$ for 6 months

43 times per month.

Suppose f is increasing function on $a \le x \le b$.

Left RS underestimate

Right RS overestimate

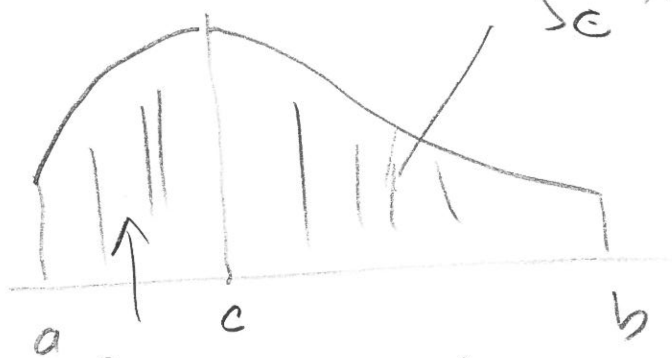
$$(\text{Right RS} - \text{Left RS}) = (f(b) - f(a)) \cdot \Delta x \qquad \Delta x = \frac{b-a}{n}$$

As $n \to \infty$, we see gap between Right RS and Left RS $\to 0$ so there is a limit. So $\lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \, f(x_i)$ <u>exists</u>.

<u>Conclude</u> Any function which is <u>increasing is integrable</u>

Same for de<u>creasing</u> functions

$\int_c^b f(x)\,dx$ exists because f decreases.



$\int_a^c f(x)\,dx$ exists because f increasing