Definition of definite integral f is a function on interval $a \le x \le b$

We say f is <u>integrable</u> on $[a,b]$ if the Riemann Sums

$$\sum_{i=1}^{n} \underbrace{f(x_i)\left(\frac{b-a}{n}\right)}$$

area of rectangle height $f(x_i)$, base $\left(\frac{b-a}{n}\right)$

have a limit as $n \to \infty$ (for <u>any</u> choice of x_i in our RS).

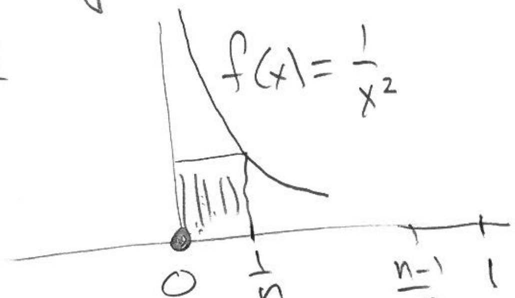
The value is called the <u>definite integral of f</u> and we denote it by $\int_a^b f(x)\,dx$.

Example $f(x) = x^2$ on $0 \le x \le b$. $\lim_{n\to\infty} RS = \frac{b^3}{3}$ so

$$\int_0^b x^2\,dx = \frac{b^3}{3}.$$

Recall if f is increasing / decreasing on ALL of $a \le x \le b$, then f is integrable.

<u>Example</u> $f(x) = \frac{1}{x^2}$



$$f(x) = \begin{cases} 0 & \text{for } x = 0. \\ \frac{1}{x^2} & \text{for } 0 < x \le 1 \end{cases}$$

decreasing except at 0.

Take n, and Right RS. 1st term of Right RS.

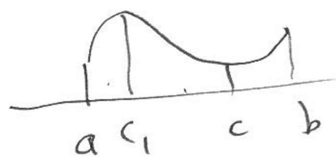
$$\underbrace{\left(\frac{1}{n}\right)}_{\text{base}}\cdot f\left(\frac{1}{n}\right) = \left(\frac{1}{n}\right)\frac{1}{\left(\frac{1}{n}\right)^2} = n$$

So Right RS $> n$. So $\lim_{n\to\infty}$ Right RS $\geq \lim_{n\to\infty} n = \infty$.

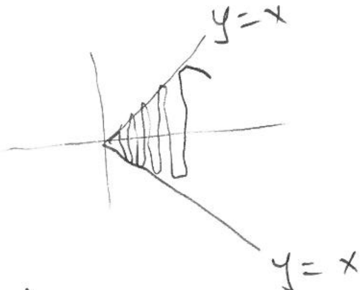
The function $f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x^2} & 0 < x \leq 1 \end{cases}$ is NOT integrable on $0 \leq x \leq 1$

Good cases where function is integrable.

(1) Can partition $a \leq x \leq b$ into finite number of intervals where f is only increasing or only decreasing

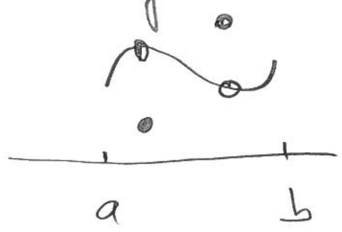


(2) f continuous. Ex $f(x) = \begin{cases} 0 & x = 0 \\ x\sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \end{cases}$



is continuous (so integrable) but there are ∞ many subintervals where f is increasing and ∞ many subintervals ——— decreasing.

Remark. If f is integrable on $a \le x \le b$.

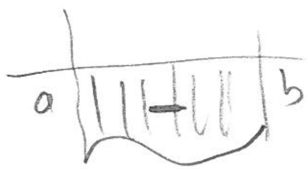


and we change the value at finitely many inputs, the new function is also integrable.

Basic properties/facts of definite integrals

① If $f \geq 0$ on interval $a \le x \le b$, then $\int_a^b f(x)\,dx$ represents/is the area between graph and x-axis

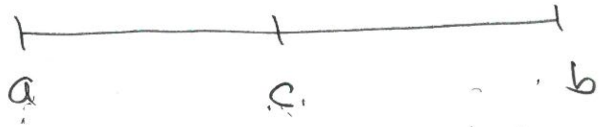
If $f \le 0$ on interval, then $\int_a^b f(x)\,dx = -$ area



② If f, g are both integrable on $a \le x \le b$, and K, L are constants. Then combo $Lf(x) + Kg(x)$ is also integrable and

$$\int_a^b (Lf(x) + Kg(x))\,dx = L\left(\int_a^b f(x)\,dx\right) + K\left(\int_a^b g(x)\,dx\right).$$

③ Additive property

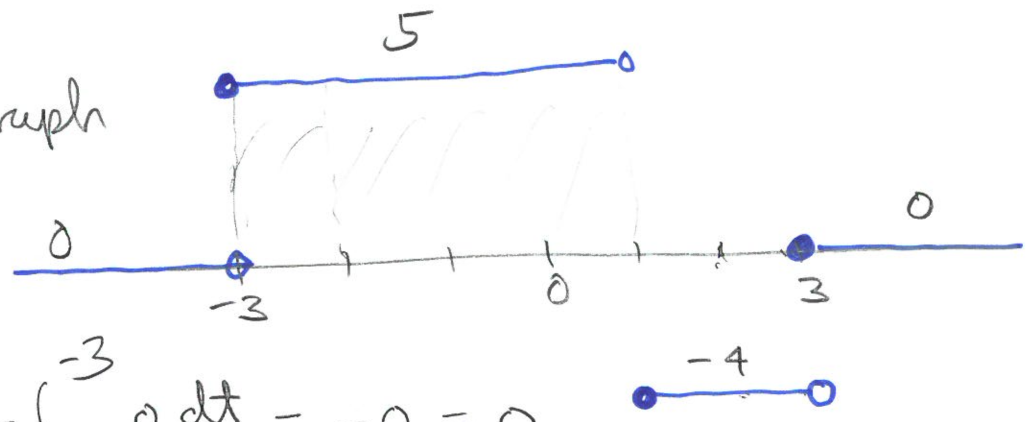


If f is integrable on $a \le x \le b$, then integrable on $a \le x \le c$ as well as $c \le x \le b$. And vice versa

And then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

<u>Convention</u> If $b \le a$, we $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

WW9 #2 $f(x)$ given by graph



Set $g(x) = \int_{-3}^{x} f(t)\,dt$

(a) Find $g(-7) = \int_{-3}^{-7} 0\,dt = -\int_{-7}^{-3} 0\,dt = -0 = 0$

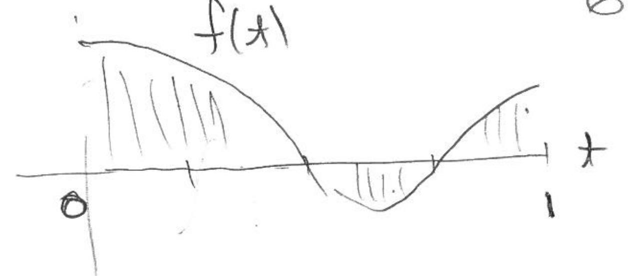
(b) $g(-2) = \int_{-3}^{-2} 5\,dt = 1 \cdot 5 = 5$

(c) $g(2) = \int_{-3}^{2} f(t)\,dt = \int_{-3}^{1} 5\,dt + \int_{1}^{2} -4\,dt = 4 \cdot 5 + 1 \cdot (-4)$
$= 20 - 4 = 16$

(d) $g(4) = \int_{-3}^{2} f(t)\,dt + \int_{2}^{3} f(t)\,dt + \int_{3}^{4} f(t)\,dt$
$= 16 + \int_{2}^{3} -4\,dt + \int_{3}^{4} 0\,dt = 16 + (-4) \cdot 1 + 0 = 12$

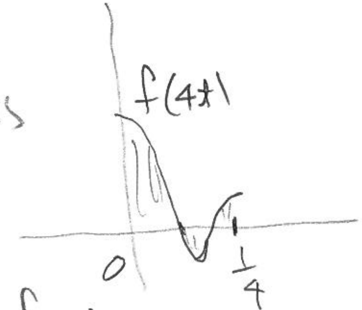
(e) Where does g have maximum value? Want just + area.
$g(1)$. And $g(1) = \int_{-3}^{1} 5\,dt = 5 \cdot 4 = 20$.

#6 Suppose $\int_0^1 f(t) = 7$. Calculate



(a) $\int_0^{1/4} f(4t)\,dt$. ~~Can be done via substitution.~~

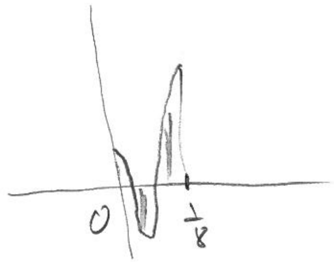
we do just by thinking areas



we have compressed horizontal by factor of 4.

So $\int_0^{1/4} f(4t)\,dt = 7 \cdot \left(\frac{1}{4}\right)$.

(b) $\int_0^{1/8} f(1-8t)\,dt = 7 \cdot \frac{1}{8} \cdot 1 = \frac{7}{8}$ | compression of horizontal by 8.

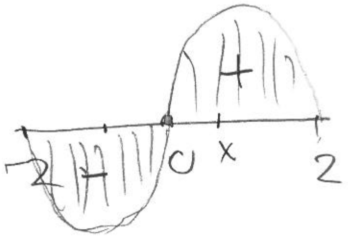


(c) $\int_{2/10}^{3/10} f(3-10t)\,dt = 7 \cdot \underset{\substack{\uparrow \\ \text{compress horizonta}}}{\frac{1}{10}} \cdot \overset{\substack{\text{flip} \\ \downarrow}}{1} = \frac{7}{10}$.

#7 Evaluate $\int_{-2}^{2} (x+8)\sqrt{x^2-4}\, dx$.

$$\int_{-2}^{2} (x+8)\sqrt{x^2-4}\, dx = \underbrace{\int_{-2}^{2} x\sqrt{x^2-4}\, dx}_{0} + 8\int_{-2}^{2} \sqrt{x^2-4}\, dx$$

Note x is odd function, $\sqrt{x^2-4}$ is even function $\Big\}$ so $x\sqrt{x^2-4}$ is odd



$$\int_{-2}^{2} \text{odd}\, dx = 0$$

$y = \sqrt{x^2-4}$ so $x^2+y^2=4$

$$\int_{-2}^{2} \sqrt{x^2-4}\, dx = \text{area}$$

$= \text{area of half disk of radius } 2$

$= \frac{1}{2}\pi \cdot 2^2 = 2\pi$

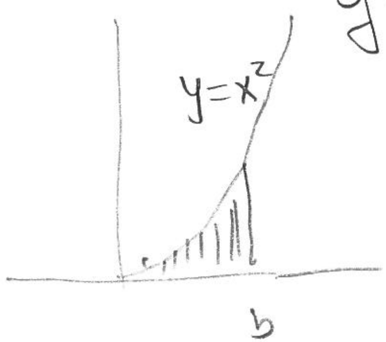
so $\int_{-2}^{2} (x+8)\sqrt{x^2-4}\, dx = 0 + 8\cdot(2\pi) = 16\pi$.

How do we find definite integral $\int_a^b f(x)\,dx$ in "general"?

Example $f(x) = x^2 \quad 0 \le x \le b$. $\int_0^b x^2\,dx = \frac{b^3}{3}$.

Note. If we think of b as variable, the function

$$g(b) = \int_0^b x^2\,dx = \frac{b^3}{3} \quad \text{(area function)}$$



has derivative $g'(b) = \frac{3b^2}{3} = b^2$.

The function/rule $b \longrightarrow b^2$ is the function we started with.