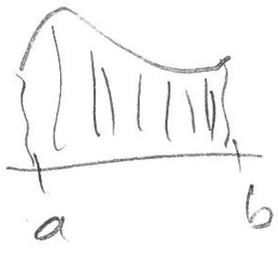
# Definition of definite integral

A function $f$ with domain $a \le x \le b$ is integrable if

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right) \quad \text{(ANY choice of } x_i \text{ in ith subinterval)}$$

exists. We denote value as $\int_a^b f(x)\,dx$



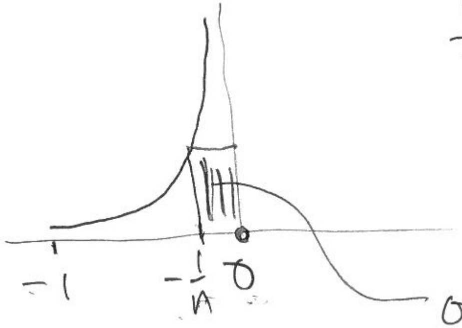

Intuition If $f \geq 0$, then $\int_a^b f(x)$ represents area (between graph and x-axis).

If $f$ is increasing on (all) of $[a,b]$, then it is integrable.

Caution Increasing must be on ALL of $[a,b]$.

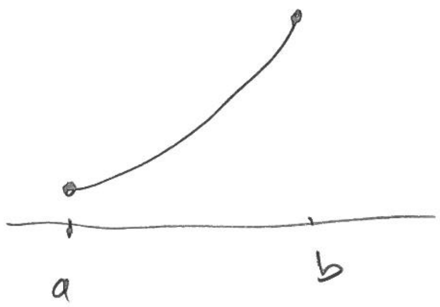
Example

$$f(x) = \begin{cases} \frac{1}{|x|^2} & \text{when } -1 \le x < 0 \\ 0 & \text{when } x = 0. \end{cases}$$



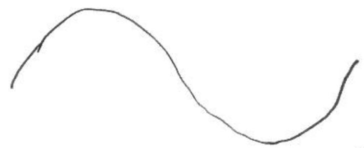

over $-\frac{1}{n} \le x \le 0$, we have $f(-\frac{1}{n}) = \frac{1}{(-\frac{1}{n})^2} = n^2$

area $= \frac{1}{n} \cdot f(-\frac{1}{n}) = \frac{1}{n} \cdot n^2 = n$



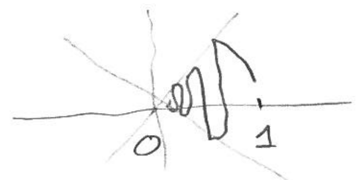

f increasing on all of $a \le x \le b$, so integrable.

(2) If we can partition $a \le x \le b$ into finite subintervals where f is only increasing or only decreasing, then f is integrable.



(3) If f is continuous on $a \le x \le b$, then integrable.

The function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } 0 < x \le 1 \\ 0 & \text{for } x = 0 \end{cases}$




is continuous on $0 \le x \le 1$ so integrable

Infinitely many subintervals where it is increasing (decreasing).

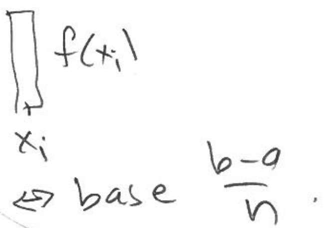
WW 8 # 9. The sum $\sum_{i=1}^{n} \frac{1}{1+i\left(\frac{7}{n}\right)}\left(\frac{7}{n}\right)$ is the <u>Right</u> R.S.

for definite integral $\int_1^b f(x)\,dx$.

Identify interval $[a,b]$ and the function.

If we divide interval in $n$ equal subintervals, a RS looks like

$$\sum_{i=1}^{n} f(x_i)\boxed{\frac{(b-a)}{n}}$$




We match $\frac{7}{n}$ to $\frac{b-1}{n}$, tells us $b=8$ so our interval is $1 \le x \le 8$.

The right end point of $i$th subinterval is $1+i\left(\frac{7}{n}\right)$. We must have


$$f\left(1+i\left(\tfrac{7}{n}\right)\right) = \frac{1}{1+i\left(\frac{7}{n}\right)}$$

So $f$ is $\frac{1}{x}$. Interval is <u>$1 \le x \le 8$</u>, function $f(x) = \frac{1}{x}$.

#10 Find RS for $\int_2^6 \frac{x}{1+x^5}\,dx$

$$RS = \sum_{i=1}^{n} f(x_i)\left(\frac{4}{n}\right) \qquad f(x) = \frac{x}{1+x^5}$$

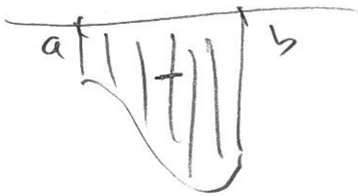
Right endpoint of ith subinterval $2+i\left(\frac{4}{n}\right)$, so $f\left(2+i\left(\frac{4}{n}\right)\right) = \dfrac{2+i\left(\frac{4}{n}\right)}{1+\left(2+i\left(\frac{4}{n}\right)\right)^5}$

$$RS = \sum_{i=1}^{n} \left(\frac{4}{n}\right) \frac{2+i\left(\frac{4}{n}\right)}{1+\left(2+i\left(\frac{4}{n}\right)\right)^5}$$

# Basic properties/facts of definite integral

① If $f \geq 0$ on $a \leq x \leq b$, then $\int_a^b f(x)\,dx$ represents <u>area</u>.

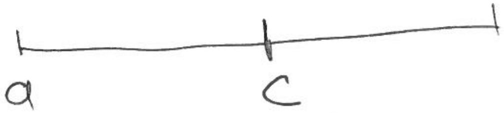
If $f \leq 0$ on $a \leq x \leq b$, then $\int_a^b f(x)\,dx$ is minus area




② <u>Linearity</u> If $f, g$ are integrable on $a \leq x \leq b$, and $K, L$ are <u>constants</u>, then the function $Kf + Lg$ is also integrable and

$$\int_a^b \left(Kf(x) + Lg(x)\right)dx = K\int_a^b f(x)\,dx + L\int_a^b g(x)\,dx.$$

③ Additive.

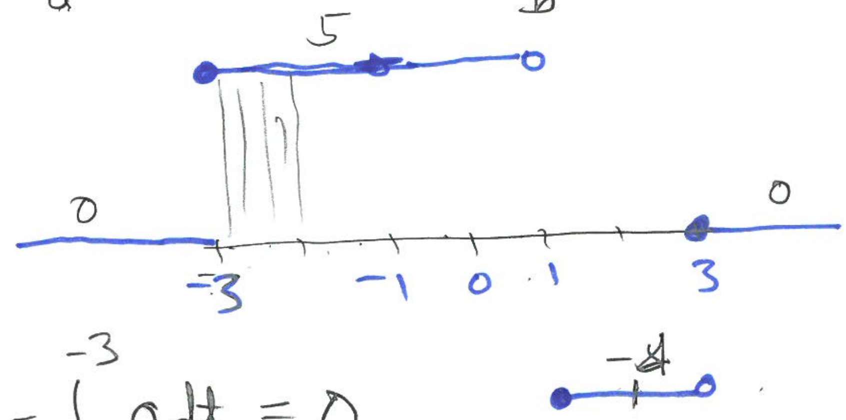



Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

**Convention** If $b \le a$ we set $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

WW9 #2 $f(x) =$ as shown.




$$g(x) = \int_{-3}^{x} f(t)\,dt$$

(a) $g(-7) = \int_{-3}^{-7} f(t)\,dt = \int_{-7}^{-3} f(t)\,dt = \int_{-7}^{-3} 0\,dt = 0.$

(b) $g(-2) = \int_{-3}^{-2} f(t)\,dt = \int_{-3}^{-2} 5\,dt = 5 \cdot 1 = 5$

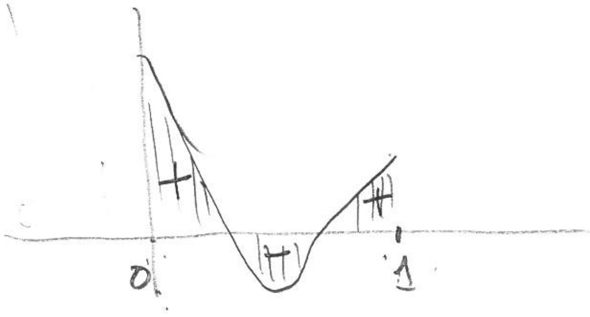
(c) $g(2) = \int_{-3}^{2} f(t)\,dt = \int_{-3}^{1} 5\,dt + \int_{1}^{2} -4\,dt = 5 \cdot 4 + (-4) \cdot 1 = 16.$

(d) $g(4) = \int_{-3}^{2} f(t)\,dt + \int_{2}^{3} f(t)\,dt + \int_{3}^{4} f(t)\,dt$

$= 16 + \int_{2}^{3} -4\,dt + \int_{3}^{4} 0\,dt = 16 - 4 + 0 = 12.$

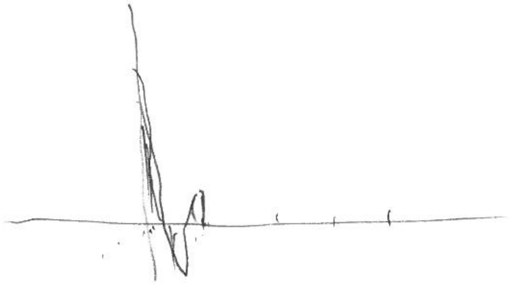
(e) Function $g(x)$ has max at $x = 1$ (max area) with value $4 \cdot 5 = 20.$

#6 Suppose $f$ integrable function with $\int_0^1 f(t)\,dt = 7$.




Find each following

(a) $\int_0^{1/4} f(4t)\,dt$ we have "compressed" in the horizontal direction by a factor of $\underline{4}$



so $\int_0^{1/4} f(4t)\,dt = \frac{7}{4}$.

(b) $\int_0^{1/8} f(1-8t)\,dt$. horizontal compression by factor of $\underline{8}$ and a $\underline{\text{reflection}}$

$$\int_0^{1/8} f(1-8t)\,dt = 7 \cdot \left(\frac{1}{8}\right) \cdot 1 = \frac{7}{8}.$$

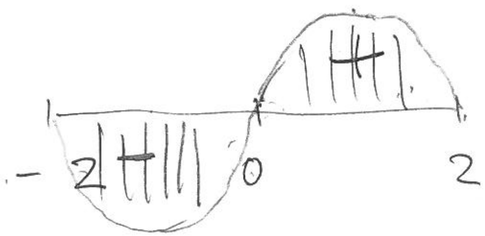
(c) $\int_{2/10}^{3/10} f(3-10t)\,dt$ horizontal compression by factor of 10 and a reflection

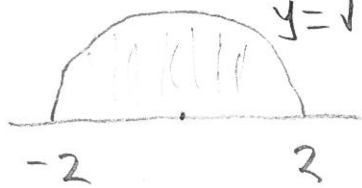
$$= 7 \cdot \left(\frac{1}{10}\right) \cdot 1 = 7/10.$$

#7 Evaluate $\int_{-2}^{2} (x+8)\sqrt{4-x^2}\,dx$

$$= \int_{-2}^{2} \underbrace{x}_{\text{odd}}\underbrace{\sqrt{4-x^2}}_{\text{even}}\,dx + \int_{-2}^{2} 8\sqrt{4-x^2}\,dx$$

$$= \int_{-2}^{2} \text{odd function}\,dx + 8\int_{-2}^{2} \sqrt{4-x^2}\,dx$$




$y=\sqrt{4-x^2}$ $\quad x^2+y^2=4$

$$= 0 + 8\int_{-2}^{2} \sqrt{4-x^2}\,dx = 8 \cdot \text{area of half disk of radius } 2.$$

$$= 8\cdot\frac{1}{2}\pi 2^2 = 16\pi.$$