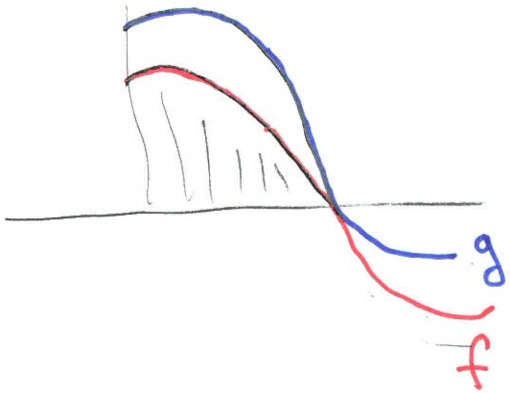
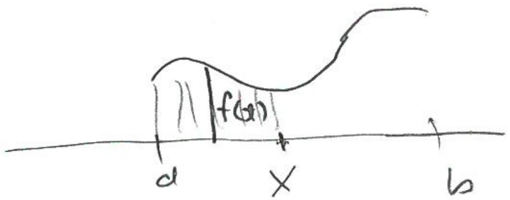
Suppose f, g are integrable functions on $a \le x \le b$, and $f(x) \le g(x)$ for all x



$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

If f is integrable on interval $a \le x \le b$, then it is also integrable on any subinterval. We define the area function as

$$A(x) = \int_a^x f(t)\,dt$$

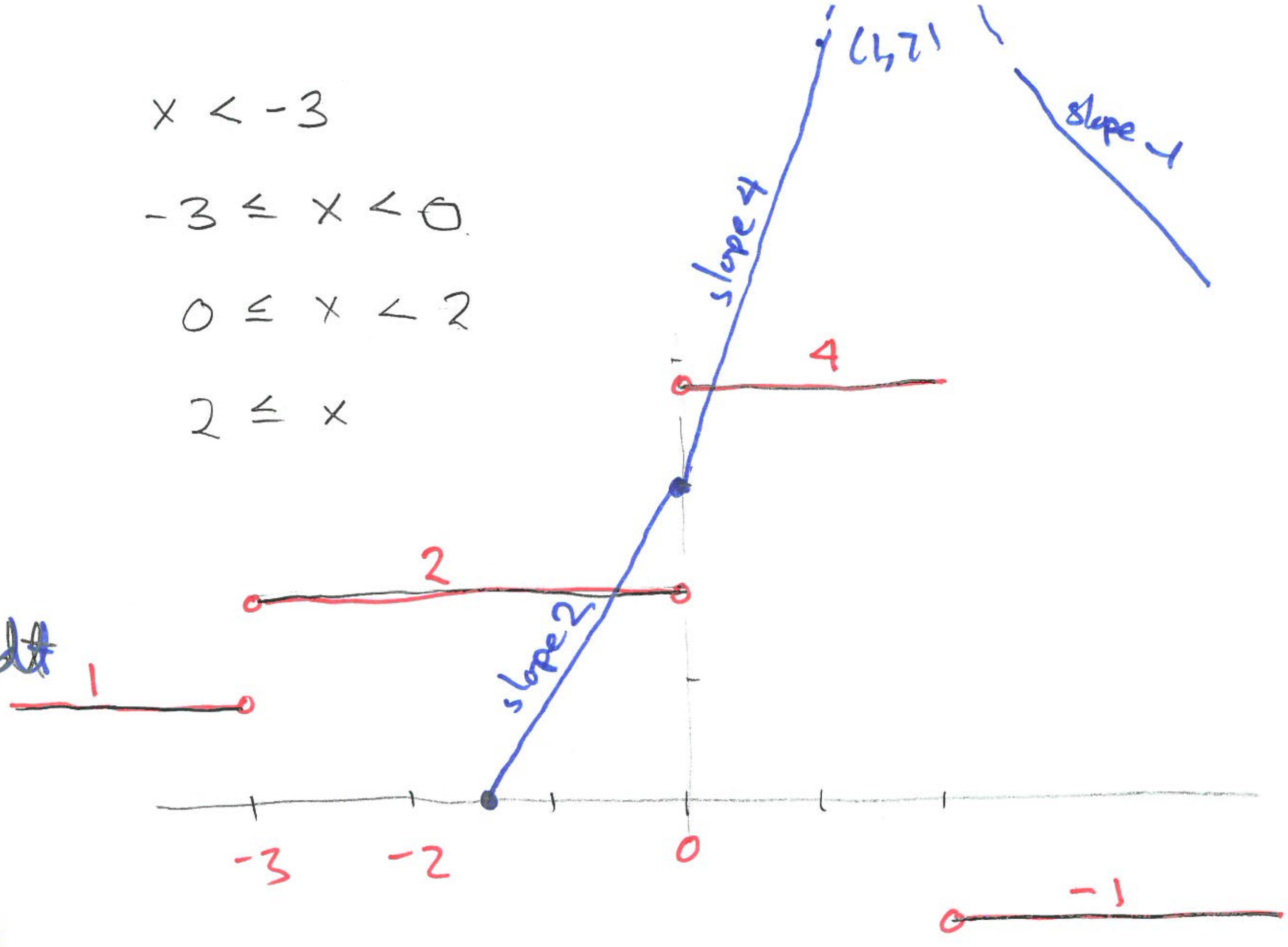


Example.

$$f(x) = \begin{cases} 1 & x < -3 \\ 2 & -3 \le x < 0 \\ 4 & 0 \le x < 2 \\ -1 & 2 \le x \end{cases}$$

Pick $a = -1.5$

$$A_{-1.5}(x) = \int_{-1.5}^{x} f(t)\,dt$$



$$A_{-1.5}(-1.5) = \int_{-1.5}^{-1.5} f(t)\,dt = 0$$

$$A_{-1.5}(0) = \int_{-1.5}^{0} f(t)\,dt = \int_{-1.5}^{0} 2\,dt = 3$$

$$A_{-1.5}(2) = \int_{-1.5}^{2} f(t)\,dt = \int_{-1.5}^{0} 2\,dt + \int_{0}^{2} 4\,dt = 3 + 8 = 11$$

Note $A'_{-1.5}$ does NOT exist where f is NOT continuous.

We note:

(i) Area function is differenticlele where f is continuous and in fact $A_{-1.5}'(x) = f(x)$

(ii) Area function is NOT differentiable where f has (jumps) discontinuity.

<u>Fundamental Theorem of Calculus I</u> If f is <u>continuous</u> on an interval I, and we look at the area function

$$A_a(x) = \int_a^x f(t)\,dt,$$

Then A_a is an <u>antiderivative</u> of f.

$A_{-2}(x)$ vs $A_{-1}(x)$

$$A_{-2}(x) = \int_{-2}^{x} f(t)\,dt = \int_{-2}^{-1} f(t)\,dt + \int_{-1}^{x} f(t)\,dt$$

$$A_{-1}(x) = \int_{-1}^{x} f(t)\,dt \qquad \text{So } A_{-2}(x) = A_{-1}(x) + \int_{-2}^{-1} f(t)\,dt$$

This matches well with the fact that $A_{-2}'(x) = f(x)$, $A_{-1}'(x) = f(x)$

Example WW9 #3 Particle moves with velocity

$$v(t) = t^2 - 3t + 2$$

Find displacement (net distance covered) over time interval $-2 \le t \le 5$

displacement = (velocity) · (time interval)

$$dp = v(t)\,dt$$

So net distance moved is $\int dp = \int_{-2}^{5} v(t)\,dt = \int_{-2}^{5} (t^2 - 3t + 2)\,dt$

Let $A(x) = \int_{-2}^{x} (t^2 - 3t + 2)\,dt$. We want $A(5)$.

FTC I $A'(x) = \cancel{v}(x) = x^2 - 3x + 2$. So $A(x) = \frac{x^3}{3} - 3\frac{x^2}{2} + 2x + C$.

constant

Now figure out C, then plug in $x = 5$.

To figure out C, we use $0 = A(-2) = \frac{(-2)^3}{3} - \frac{3(-2)^2}{2} + 2(-2) + C$.

so $C = -\left(\frac{(-2)^3}{3} - \frac{3(-2)^2}{2} + 2(-2)\right)$

So $A(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x - \left(\frac{(-2)^3}{3} - \frac{3(-2)^2}{2} + 2(-2) \right)$

Plug in 5 to get

net displacement $= \displaystyle\int_{-2}^{5} (t^2 - 3t + 2)\,dt = \left(\frac{5^3}{3} - \frac{3 \cdot 5^2}{2} + 2 \cdot 5 \right) - \left(\qquad \right)$

Fundamental Theorem of Calculus II If f is a continuous function on the interval $a \le x \le b$, and F is an antiderivative, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Notation we write this as $F(x)\Big|_a^b$

Example WW9 #4 Find $\displaystyle\int_4^5 \frac{2x^2+7}{\sqrt{x}}\,dx$

$$\int_4^5 \left(2x^{3/2} + 7x^{-1/2}\right) dx = 2\frac{x^{5/2}}{(5/2)} + 7\frac{x^{1/2}}{(1/2)} \Bigg|_4^5$$

$$= \left(\frac{4}{5}(5^{5/2}) + 14\,5^{1/2}\right) - \left(\frac{4}{5}4^{5/2} + 7\,4^{1/2}\right)$$

#5 Evaluate $\int_0^1 \frac{2e^{6x}-2}{e^{2x}}\,dx$

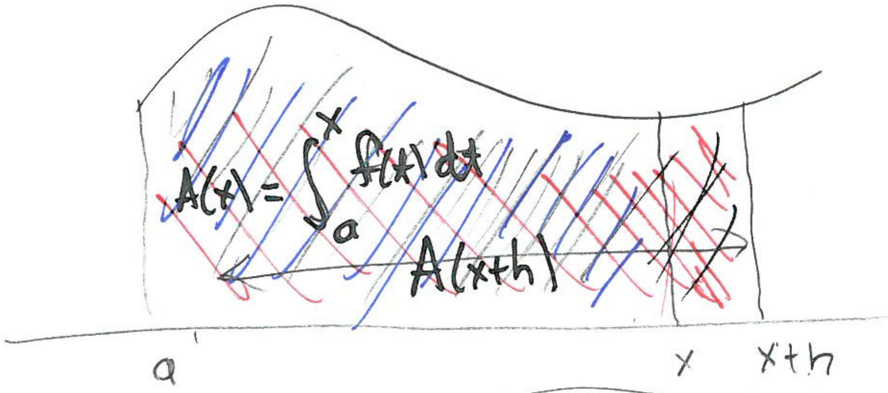
$$\int_0^1 \left(2e^{4x} - 2e^{-2x}\right) dx = \left(2\frac{e^{4x}}{4} + e^{-2x}\right)\Bigg|_0^1$$

$$= \frac{1}{2}\left(e^4 - e^0\right) + \left(e^{-2} - e^0\right)$$

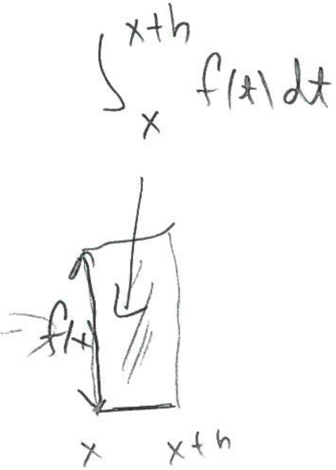
$$= \frac{1}{2}e^4 - \frac{1}{2} + e^{-2} - 1 = \frac{1}{2}e^4 + e^{-2} - \frac{3}{2}.$$

Proof of FTC I

f continuous



Derivative of area function: $\lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$



$$= \frac{\int_x^{x+h} f(t)\,dt}{h} \doteq \frac{f(x) \cdot h}{h} = f(x)$$

That f is continuous means. As $h \to 0$, we get $\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$

$$= f(x)$$

So A has derivative equal to f.