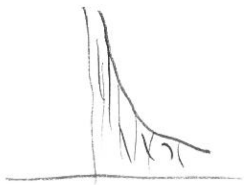
Some examples of non integrable functions

(1) $f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & 0 < x \leq 1 \end{cases}$



f is not integrable. The area between graph and x-axis in infinite.

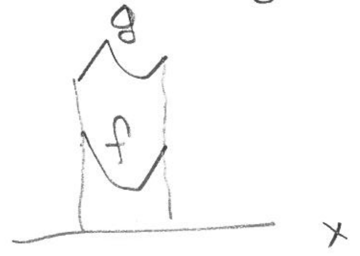
(2) For f the function $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational and } 0 \leq x \leq 1 \\ 0 & \text{if } x \text{ is irrational and } 0 \leq x \leq 1 \end{cases}$

This function is not integrable

Additional basic fact about definite integrals.

If f, g are integrable on $a \le x \le b$, and $f \le g$ everywhere

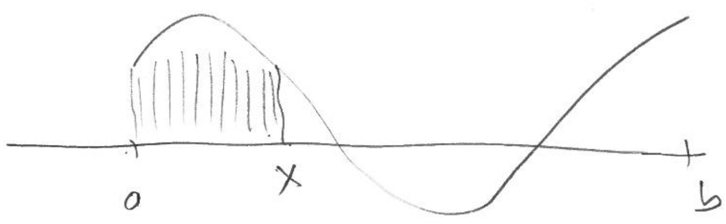
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$



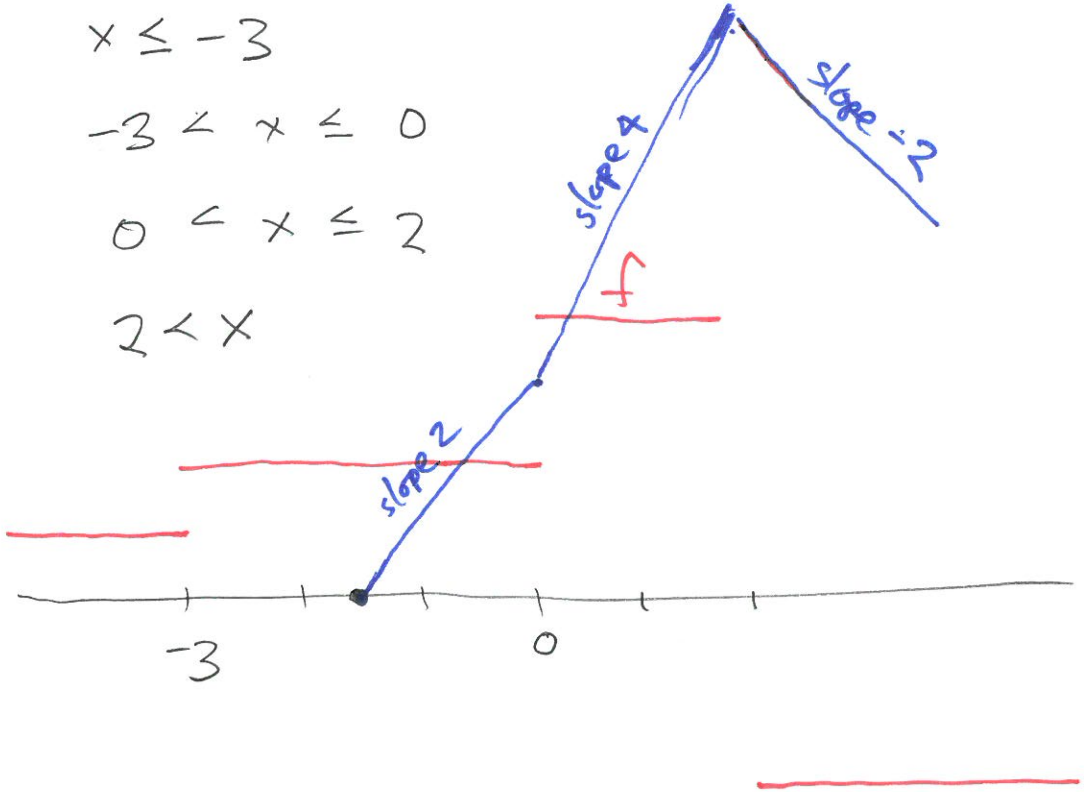
If f is integrable $a \le x \le b$, we define the "area" function

$$A(x) = \int_a^x f(t)\,dt$$

this is remind us the input variable



Example $f(x) = \begin{cases} 1 & x \le -3 \\ 2 & -3 < x \le 0 \\ 4 & 0 < x \le 2 \\ -2 & 2 < x \end{cases}$



Take $a = -1.5$

$$A(x) = \int_{-1.5}^{x} f(t)\,dt$$

$$A(-1.5) = \int_{-1.5}^{-1.5} f(t)\,dt = 0$$

$$A(0) = \int_{-1.5}^{0} f(t)\,dt = \int_{-1.5}^{0} 2\,dt = 3.$$

$$A(-3) = \int_{-1.5}^{-3} f(t)\,dt = \int_{-1.5}^{-3} 2\,dt = -3$$

Note. Area is differentiable where f was constant (continuous) and derivative equaled f.
not differentiable where f has jump (not continuous)

$A_{-3}(x) = \int_{-3}^{x} f(t)\,dt$ $\qquad$ $A_{-2}(x) = \int_{-2}^{x} f(t)\,dt$

$A_{-3}(-3) = 0$.

$A_{-3}(-1) = \int_{-3}^{-1} f(t)\,dt$

A_{-3}, A_{2} have "same graph" shifted vertically.

$$A_{-3}(x) = \int_{-3}^{x} f(t)\,dt = \int_{-3}^{-2} f(t)\,dt + \int_{-2}^{x} f(t)\,dt$$

$$= \text{constant} + A_{-2}(x).$$

The fact that these two area functions differ by a constant, means they have same derivative.

Fundamental Theorem of Calculus I

If f is continuous function on an interval I

and we consider an area function

$$A_a(x) = \int_a^x f(t)\,dt.$$

then A_a is differentiable and $A_a'(x) = f(x)$ so $A_a' = f$.

This is KEY for understanding relationship between definite integrals and antiderivatives.

Suppose we just want to find $\int_a^b f(t)\,dt$ (assume f continuous)

Set $A_a(x) = \int_a^x f(t)\,dt$. We know (1) $A_a(a) = 0$

(2) FTC I, $A_a' = f$.

If we can find antiderivative F of f, then

A_a and F differ by a constant, and (1) allows to find the constant

Once we know A_a, we have $\int_a^b f(t)\,dt = A_a(b)$.

Example. WW 9 # 3

Velocity of particle is given as $v(t) = t^2 - 3t + 2$.
How far has particle moved during time interval $-2 \le t \le 5$.

$$\text{displacement} = \text{velocity} \cdot \text{time}$$

$$dp = v(t)\, dt$$

So amount moved is $\displaystyle\int_{-2}^{5} v(t)\, dt = \int_{-2}^{5} (t^2 - 3t + 2)\, dt$

Look at area function $A_{-2}(x)$. We know

$A_{-2}(-2) = 0$

$\Rightarrow A_{-2}'(t) = t^2 - 3t + 2$.

$\frac{8}{3} + 6 + 4$

So ② $\Rightarrow$ $A_{-2}(t) = \dfrac{t^3}{3} - \dfrac{3t^2}{2} + 2t + C$.

① $\Rightarrow$ $0 = A_{-2}(-2) = -\dfrac{8}{3} - \dfrac{12}{2} + 4 + C$ so $C = 4\frac{2}{3}$

So $A_{-2}(t) = \dfrac{t^3}{3} - \dfrac{3t^2}{2} + 2t + 12\frac{2}{3}$, So $A_{-2}(5) = \left(\dfrac{5^3}{3} - \dfrac{3 5^2}{2} + 2 \cdot 5\right) + 4\dfrac{2}{3}$.

Fundamental Theorem of Calculus II (good for practical calculation)

If f is continuous on an interval $a \le x \le b$, and F is any antiderivative of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Example WW9 #4 Find $\displaystyle\int_4^5 \frac{2x^2+7}{\sqrt{x}}\,dx$

$$\int_4^5 \left(2x^{3/2} + 7x^{-1/2}\right)dx = \left(2\frac{x^{5/2}}{5/2} + 7\frac{x^{1/2}}{1/2}\right)\Bigg|_4^5 \quad \text{means}$$

$$= \left(2\frac{5^{5/2}}{5/2} + 7\frac{5^{1/2}}{1/2}\right) - \left(2\frac{4^{5/2}}{5/2} + 7\frac{4^{1/2}}{1/2}\right)$$