Student question (old MATH 1013 final)

Find $\lim\limits_{x \to +\infty} \left(\dfrac{\sqrt{x}+2}{x+3}\right)^x$ $\left(\dfrac{\sqrt{\infty}+2}{\infty+3}\right)^\infty$ $\left(\dfrac{\infty}{\infty}\right)^\infty$.

"Look at the function an analyze what is happening"

Both $\sqrt{x}$, $x \to \infty$ as $x \to +\infty$.

We see that x goes ∞ faster than $\sqrt{x}$

Divide top/bottom by $\sqrt{x}$ to get

$$\left(\frac{\sqrt{x}+2}{x+3}\right) = \frac{1+\frac{2}{\sqrt{x}}}{\sqrt{x}+\frac{3}{\sqrt{x}}} \longrightarrow \frac{1+0}{\infty+0} \text{ as } x \to \infty.$$

So $\left(\dfrac{\sqrt{x}+2}{x+3}\right) \longrightarrow 0$ as $x \to \infty$. So $\lim\limits_{x \to +\infty} \left(\dfrac{\sqrt{x}+2}{x+3}\right)^x = 0$.

$$\left(\frac{\sqrt{x}+2}{x+3}\right)^x \longrightarrow 0 \text{ as } x \to \infty$$

Fundamental Theorem of Calculus.

Suppose f is continuous on interval $a \le x \le b$.

FTC I The area function $A(x) = \int_a^x f(t)\,dt$,

has derivative f. $A'(x) = f(x)$.

This is KEY to understanding.

FTC II If F is any antiderivative of f, then the indefinite integral $\int_a^b f(x)dx$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b \quad (= F(b) - F(a)).$$

Extremely useful CALCULATION tool.

Proof of FTC $\underline{\text{II}}$. (Same idea as the velocity problem).

Since BOTH $A(x) = \int_a^x f(t)\,dt$, and F are antiderivative to f, their difference $A - F$ is a <u>constant</u> function

We evaluate at the two point a, b (get the constant)

$$A(b) - F(b) = \text{constant} = A(a) - F(a).$$

This is $\int_a^b f(x)\,dx$. Solve for it

$$A(b) = F(b) + A(a) - F(a)$$

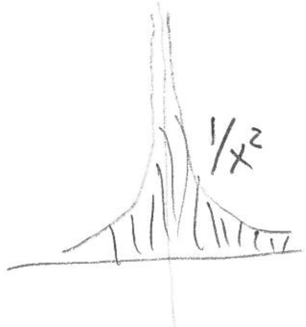
This is $\int_a^a f(t)\,dt = 0$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

CAUTION Assumption is f is continuous

BAD example

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x^2} & -1 \le x < 0 \ ; \ 0 < x \le 1 \end{cases}$$



Not continuous at 0.

Note $F(x) = \frac{-1}{x}$ is an antiderivative for $x \neq 0$.

If we use FTC II in violation assumption f is continuous

$$\int_{-1}^{1} \frac{dx}{x^2} = \left.\frac{-1}{x}\right|_{-1}^{1} = -\left(\frac{1}{1} - \frac{1}{-1}\right) = -2 \quad \text{FALSE!}$$

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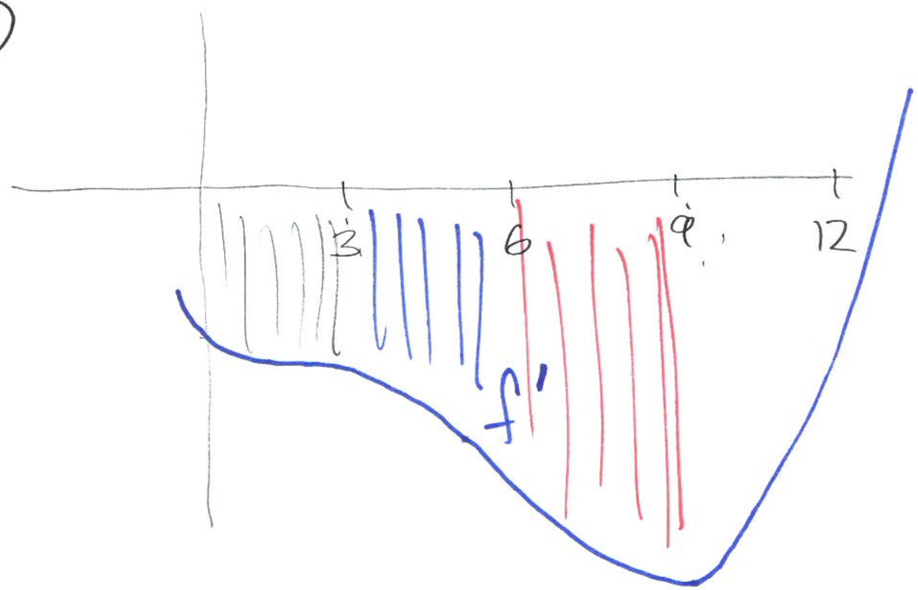
WW9 #10

GRAPH of f'



(a) Compare $f(0)$ and $f(3)$. Which is bigger?

Solution 1 Since f' is NEGATIVE, f is decreasing so $f(3) < f(0)$.

Solution 2 From graph, f' is continuous and it has f as an antider. By FTC II

$$0 > \int_0^3 \underbrace{f'(t)}_{\text{negative}}\,dt = f(3) - f(0) \quad \text{so } f(0) > f(3)$$

(b) Determine order of

$$f(9) - f(6) = \int_6^9 f'(t)\,dt \quad \text{RED AREA} \quad (\text{FTC II})$$

$$f(6) - f(3) = \int_3^6 f'(t)\,dt \quad \text{BLUE AREA}$$

Red area is more NEGATIVE than blue area so $f(9) - f(6) < f(6) - f(3)$

$$\frac{f(9) - f(3)}{2} = \frac{1}{2}\int_3^9 f'(t)\,dt = \text{AVERAGE of blue and red area}$$

so $$f(9) - f(6) < \frac{f(9) - f(3)}{2} < f(6) - f(3)$$

#11 For $F(x) = \int_{\sqrt{x}}^{1} \frac{s^2}{2+3s^4}\,ds$, find $F'(x)$.

FTC I Write $F(x) = -\int_{1}^{\sqrt{x}} \frac{s^2}{2+3s^4}\,ds$.

Set $A(x) = -\int_{1}^{x} \frac{s^2}{2+3s^4}\,dx$. Then $F(x) = A(\sqrt{x})$

This means F is composition of A and $\sqrt{x}$. To find derivative use chain rule

$$F'(x) = A'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = -\frac{(\sqrt{x})^2}{2+3(\sqrt{x})^4} \cdot \frac{1}{2}x^{-1/2}$$

FTC

$$= \frac{-x}{2+x^2}\,\frac{1}{2}x^{-1/2}.$$

If f is integrable on $a \le x \le b$,

$$\frac{\int_a^b f(x)\,dx}{b-a}$$ is called MEAN / AVERAGE value of function

Example. $v(t)$ = velocity of a particle $\int_a^b v(t)\,dt$ = net displacement

Here $\frac{\int_a^b v(t)\,dt}{b-a}$ represents AVERAGE / MEAN velocity

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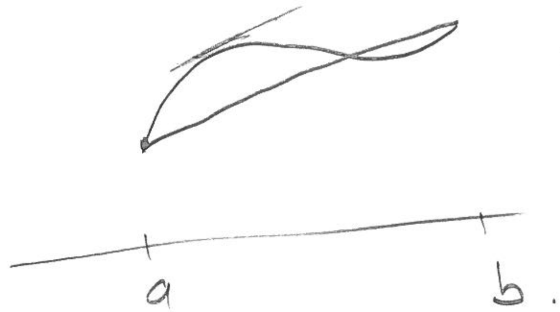
Explanation of Why Mean Value Theorem called MVT

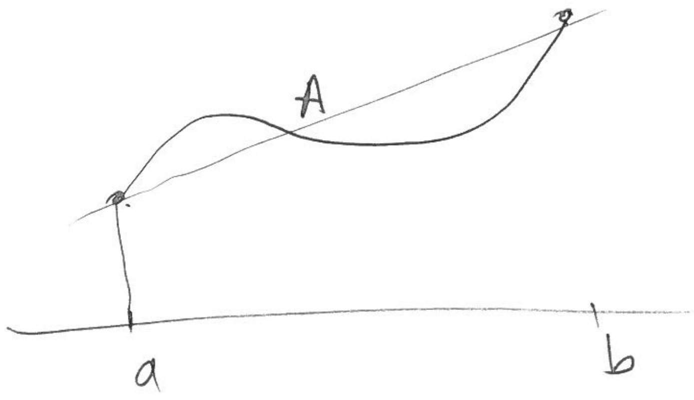


$\exists$ point c so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Suppose f is continuous on $a \le x \le b$

$$A(x) = \int_a^x f(t)\,dt$$



Secant slope from $(a, A(a))$ to $(b, A(b))$ is

$$\frac{A(b)-A(a)}{b-a} = \frac{\int_a^b f(x)dx - 0}{b-a}$$

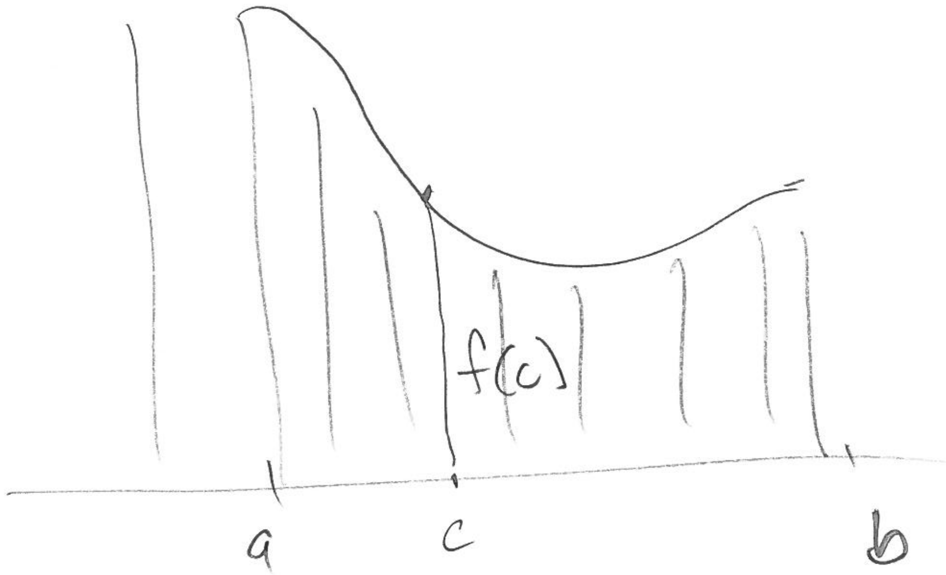
$$= \frac{1}{b-a}\int_a^b f(x)dx = \text{MEAN/AVERAGE}$$

value of the function f

MVT say there interior point c so that $\underset{\substack{\| \\ f(c)}}{A'(c)} = \frac{A(b)-A(a)}{b-a}$

So

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$



$$\frac{\int_a^b f(x)\,dx}{b-a} = \text{avg/mean value of } f$$

There is an interior point c so f(c) is the average/mean value of the function f.

Two important rules/way to find derivatives
(1) chain
(2) product.

FTC I allows us to "reverse thing".
When we find antiderivative we reverse rules (1) chain
(2) product.

Reversing chain rule is called integration by substitution
Reversing product rule for derivatives is called integration by parts.