Student question (old MATH 1013 final)

$$\lim_{x \to +\infty} \left(\frac{\sqrt{x}+2}{x+3} \right)^x \qquad \left(\frac{\infty}{\infty}\right)^\infty$$

"Look at the limit" x vs $\sqrt{x}$

As $x \to \infty$ the function $x \to \infty$ much faster than $\sqrt{x}$.

$$\left(\frac{\sqrt{x}+2}{x+3}\right) = \left(\frac{1+\frac{2}{\sqrt{x}}}{\sqrt{x}+\frac{3}{\sqrt{x}}}\right) \to \frac{1+0}{\infty+0} \to 0 \quad \Big\} \; 0^\infty$$

So $$\lim_{x \to +\infty} \left(\frac{\sqrt{x}+2}{x+3}\right)^x = \lim_{x\to\infty}\left(\frac{1+\frac{2}{\sqrt{x}}}{\sqrt{x}+\frac{3}{\sqrt{x}}}\right)^x = 0.$$

Fundament Theorem of Calculus I Suppose f is continuous on the interval $a \le x \le b$, and

$$A(x) = \int_a^x f(t)\,dt$$

UNDERSTANDING

Then $A'(x) = f(x)$ Area function is an antiderivative

Fundamental Theorem of Calculus II f continuous on $a \le x \le b$, and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \left(\text{notation } F(x)\Big|_a^b\right).$$

CALCULATION TOOL.

"Proof of FTC I" We apply derivative definition to area function

$$\frac{A(x+h) - A(x)}{h} = \frac{\int_x^{x+h} f(t)\,dt}{h} \doteq \frac{f(x) \cdot h}{h} = f(x)$$

and in limit $\displaystyle\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$

"Proof of FTC II" By FTC I, the area function A is an antiderivative of f on $a \le x \le b$. If F is another antiderivative, So the difference $A - F =$ constant.

Plug in a and b.

$$\int_a^a f(x)\,dx = 0$$

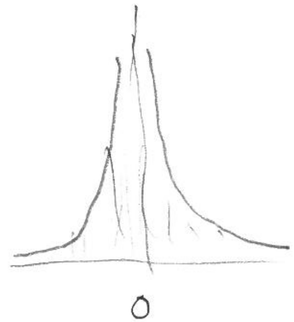
$$A(b) - F(b) = \text{constant} = A(a) - F(a)$$

$$\int_a^b f(x)\,dx = A(b) = F(b) - F(a) + \underbrace{\cancel{A(a)}}_{0} = F(b) - F(a).$$

CAUTION The function f must be continuous.

BAD Example

$$f(x) = \begin{cases} \dfrac{1}{x^2} & -1 \le x < 0, \quad 0 < x \le 1 \\ 0 & x = 0. \end{cases}$$



f is discontinuous.

Note $F(x) = -\dfrac{1}{x}$ has $F'(x) = \dfrac{1}{x^2}$ for $x \neq 0$.

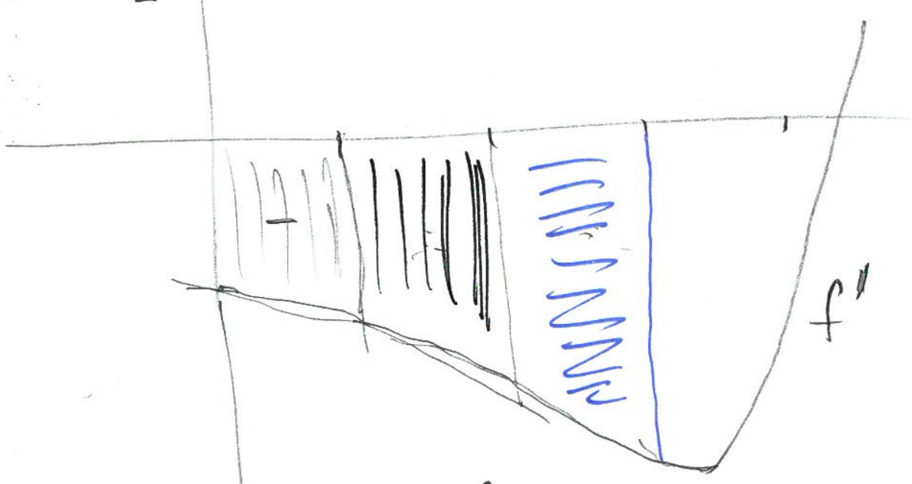
$$\int_{-1}^{1} \frac{dx}{x^2} \overset{?}{=} -\frac{1}{x}\Big|_{-1}^{1} = -\left(\frac{1}{1} - \frac{1}{-1}\right) = -2$$

↑

FALSE.

WW 9 #10 Graph of <u>derivative f'</u> is given as

$f(9)-f(6) < \frac{1}{2}(f(9)-f(3)) < f(6)-f(3)$



$|f(6)-f(3)|$

$|\frac{1}{2}(f(9)-f(3))|$ = average of

$|f(9)-f(6)|$

(a) Compare <u>f(0)</u> and <u>f(3)</u> Since on $0 \le x \le 1$, the derivative is negative. $f(3) < f(0)$.

② solution. f is an antiderivative of f' so by FTC II

$$f(3)-f(0) = \int_0^3 f'(x)\,dx < 0 \text{ since } f' \text{ negative.}$$

(b) $f(6)-f(3)$ is area in |||| (negative area) $f(9)-f(6) < f(6)-f(3)$

$f(9)-f(6)$ is area in ≡ $f(9)-f(3)$ = combined ≡ and |||| areas.

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#11 If $F(x) = \int_{\sqrt{x}}^{1} \frac{s^2}{2+3s^4}\, ds$

$\int_1^x \frac{s^2}{2+3s^4}\, ds$

directly apply FTC I

$$= -\int_1^{\sqrt{x}} \frac{s^2}{2+3s^4}\, ds.$$

$$= A(\sqrt{x}) \quad \text{where } A(t) = -\int_1^t \frac{s^2}{2+3s^4}\, ds, \quad A'(t) = -\frac{t^2}{2+3t^4}$$

To find $F'(x)$ we use FTC I and Chain rule

$$F(x) = A(\sqrt{x})$$

$$F'(x) = A'(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2} = -\frac{(\sqrt{x})^2}{2+3(\sqrt{x})^4} \cdot \frac{1}{2} x^{-1/2}$$

$$= -\frac{x}{2+3x^2} \cdot \frac{1}{2} \frac{1}{\sqrt{x}}.$$

Two extremely important rules to find derivatives are

- Chain rule $\rightsquigarrow$ + FTC I $\rightarrow$ technique of substitution
- product rule $\rightsquigarrow$ + FTC I $\rightarrow$ technique of integration by parts.

We use FTC I to "reverse" these rules to give techniques to find antiderivatives

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Technique of substitution

$y = g(x)$, $z = f(y)$, so $z = f(g(x))$

$\frac{dy}{dx} = g'(x)$ $\quad \frac{dz}{dy} = f'(y)$ $\quad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)\, g'(x)$

$$= f'(g(x))\, g'(x).$$

Reverse this

WW9 #8 Find antiderivative / indefinite integral

$$\int e^x \sqrt{4+e^x}\, dx$$

We do a transformation/substitution $u = 4+e^x$

$$\frac{du}{dx} = (0+e^x), \quad du = e^x\, dx$$

Substitute to get

$$\int e^x \sqrt{4+e^x}\, dx = \int \sqrt{4+e^x}\, \underbrace{e^x dx} = \int \sqrt{u}\, \underbrace{du}$$

$$= \frac{u^{3/2}}{3/2} + C = \frac{(4+e^x)^{3/2}}{3/2} + C.$$

#9 Find antider/indef integral

$$\int \frac{2\sin(x)}{1+\cos^2 x}\,dx$$

Let $u = \cos x$, so $\frac{du}{dx} = -\sin x$, $du = -\sin x\,dx$

$$\int \frac{2\sin(x)\,dx}{1+\cos^2 x} = -2\int \frac{-\sin x\,dx}{1+\cos^2 x} = -2\int \frac{du}{1+u^2}$$

$$= -2\arctan(u) + C$$

$\frac{1}{1+u^2}$ has antiderivative $\arctan(u)$

$$= -2\arctan(\cos(x)) + C.$$