Substitution

Reverse of chain rule

composite function $(f(g(x)))' = f'(g(x))\, g'(x)$

$z = f(y)$, $y = g(x)$ $\qquad f'(y) \quad g'(x)$

$\frac{dz}{dy} = f'(y) \quad \frac{dy}{dx} = g'(x) \qquad \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$ "dy cancel"

In finding antiderivative

$\int \boxed{f'(g(x))}\, \boxed{g'(x)\, dx} = f(g(x))$.

Example
WW9 #8

Find $\int e^x (4+e^x)^{1/2} dx = \int (4+e^x)^{1/2} \boxed{e^x dx}$

$= \frac{(4+e^x)^{3/2}}{\frac{3}{2}} + C.$

suggest $g(x) = e^x$; $g'(x) = e^x$

$f'(e^x) = (4+e^x)^{1/2}$ so f $\frac{(4+e^x)^{3/2}}{\frac{3}{2}} = f(g(x))$

Redo Find $\int e^x (4+e^x)^{1/2} dx = \int (4+e^x)^{1/2} e^x \, dx$

Let $u = 4+e^x$, so $\frac{du}{dx} = (0+e^x)$ so $du = e^x dx$

"Substitute" to get

$$\int (4+e^x)^{1/2} e^x dx = \int u^{1/2} du = \frac{u^{3/2}}{3/2} + C$$

$$= \frac{(4+e^x)^{3/2}}{3/2} + C$$

#9 Find antiderivative / indefinite integral

$$\int \frac{2\sin(x)}{1+\cos^2(x)}\,dx$$

Let $u = \cos(x)$, so $\frac{du}{dx} = -\sin(x)$, $du = -\sin(x)\,dx$

$$2\int \frac{\sin(x)}{1+\cos^2 x}\,dx = 2\int \frac{-du}{1+u^2} = -2\int \frac{du}{1+u^2}$$

Since $(\arctan u)$ has derivative $\frac{1}{1+u^2}$ we have

$$-2\int \frac{du}{1+u^2} = -2\arctan(u) + C = -2\arctan(\cos(x)) + C$$

#12 Find definite integral $\int_0^1 \frac{dx}{\sqrt{x} + 4\sqrt[3]{x}}$

Want substitution to get rid of $\sqrt{x}$ and $\sqrt[3]{x}$.

Let $x = u^6$, so $\frac{dx}{du} = 6u^5$ so $dx = 6u^5\,du$

$\sqrt{x} = (u^6)^{1/2} = u^3$, $\sqrt[3]{x} = (u^6)^{1/3} = u^2$

$x=0$, $u=0$

$x=1$, $u=1$

So substitution

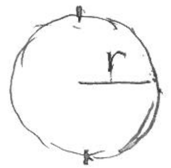
$$\int_{x=0}^{x=1} \frac{dx}{\sqrt{x}+4\sqrt[3]{x}} = \int_{u=0}^{u=1} \frac{6u^5\,du}{u^3+4u^2} = 6\int_0^1 \frac{u^3\,du}{u+4}$$

$$= 6\int_0^1 (u^2 - 4u + 16) + \frac{-64}{u+4}\,du$$

$$= 6\left\{ \frac{u^3}{3} - 2u^2 + 16u - 64\ln(u+4) \Big|_0^1 \right\}.$$

$$\begin{array}{r|l}
 & u^2 - 4u + 16 \\ \hline
u+4 & u^3 + 0u^2 + 0u^1 + 0 \\
 & u^3 + 4u^2 \\ \hline
 & -4u^2 \\
 & -4u^2 - 16u \\ \hline
 & +16u \\
 & 16u + 64 \\ \hline
 & -64
\end{array}$$

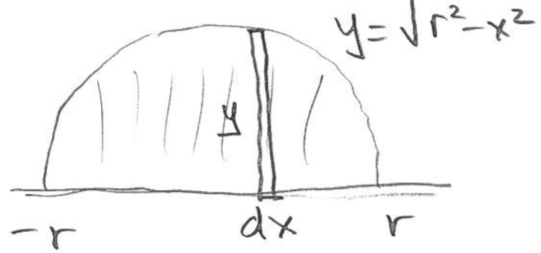
Area of a circle/disk of radius r.

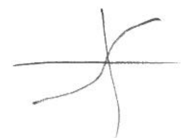


$\pi = \dfrac{\text{circumference}}{\text{dia}}$ $\quad$ circum $= \pi 2r$.

Area of circle of radius r is $\underline{A = \pi r^2}$. "Not proved in elementary school"

<u>Proof</u>





$$\text{half area} = \int_{-r}^{r} y\,dx = \int_{x=-r}^{x=r} \sqrt{r^2-x^2}\,dx$$

Use substitution.

$x = r\sin\theta$

$dx = (r\cos\theta)\,d\theta$

when $x=-r \longleftrightarrow \theta = \frac{-\pi}{2}$

$x = r \longleftrightarrow \theta = \frac{\pi}{2}$

$$= \int_{\theta=\frac{-\pi}{2}}^{\theta=\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\, d\theta$$

$$= r^2 \int_{-\pi/2}^{\pi/2} (\cos\theta)^2\, d\theta.$$

Use trig identity

$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$$\cos(2\theta) = (\cos\theta)^2 - (\sin\theta)^2 = (\cos\theta)^2 - (1-(\cos\theta)^2)$$
$$= 2\cos^2\theta - 1$$

$$(\cos\theta)^2 = \frac{\cos(2\theta)+1}{2}$$

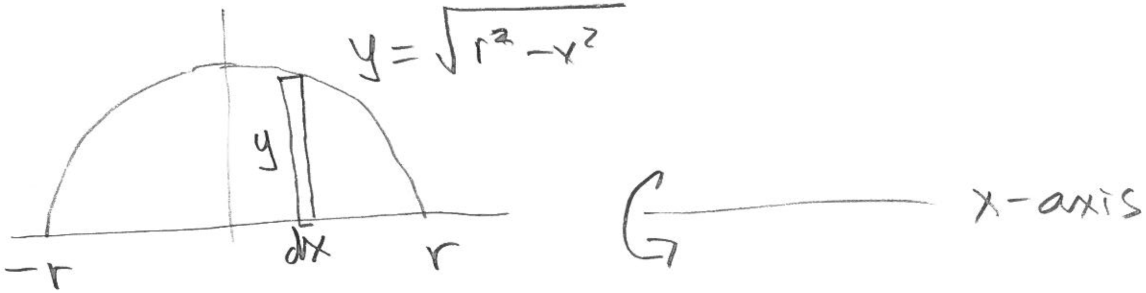
half area $= r^2 \int_{-\pi/2}^{\pi/2} \frac{\cos(2\theta)+1}{2}\, d\theta = \frac{r^2}{2} \int_{-\pi/2}^{\pi/2} (\cos 2\theta) + 1 \, d\theta$

$$= \frac{r^2}{2} \left(\frac{\sin(2\theta)}{2} + \theta \Big|_{-\pi/2}^{\pi/2} \right)$$

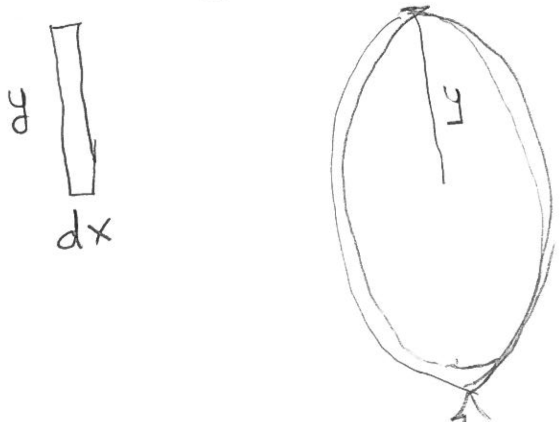
$$= \frac{r^2}{2} \left(\left(0 + \frac{\pi}{2}\right) - \left(0 - \frac{\pi}{2}\right)\right) = \frac{\pi r^2}{2}$$

So area of disk radius r is πr^2.

Volume of sphere



We get sphere by rotating half disk about x-axis



thickness of "coin" is dx

$(\pi y^2) \cdot dx$

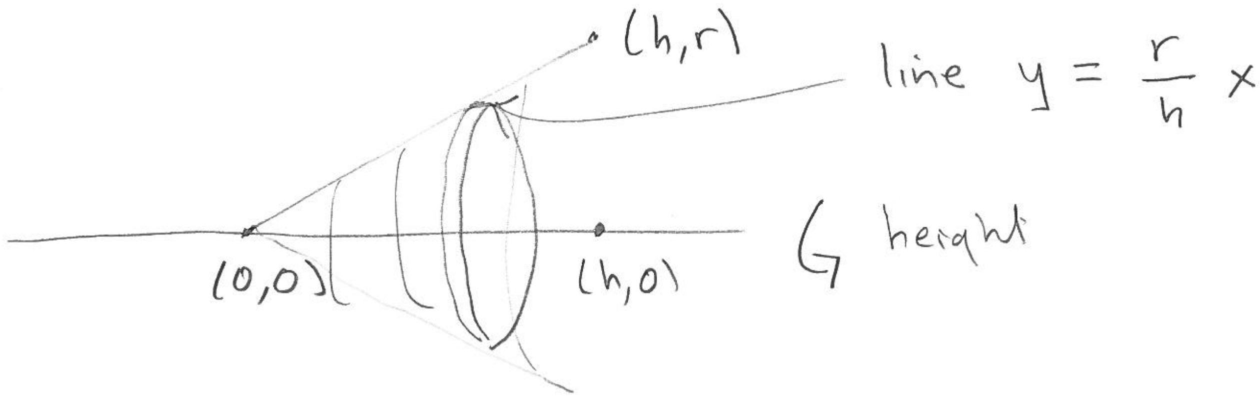
volume of "coin" is $\underset{\text{base}}{\text{Area}} \cdot \underset{\text{height}}{dx}$

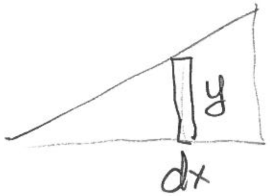
$$V = \int_{-r}^{r} \boxed{\pi y^2 \cdot dx} = \int_{-r}^{r} \pi (r^2 - x^2)\, dx$$

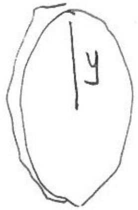
$$= \pi \int_{-r}^{r} (r^2 - x^2)\, dx = \pi \left\{ r^2 x - \frac{x^3}{3} \Big|_{-r}^{r} \right\}$$

$$= \pi \left\{ \left(r^3 - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right) \right\} = \frac{4\pi}{3} r^3 .$$

Volume of cone base radius r, height h.







thickness dx

volume = area base $\cdot$ thickness

$= \pi y^2 \cdot dx$

$$\frac{h^3}{3}$$

$$\int_0^h \pi y^2 \, dx = \int_0^h \pi \left(\frac{r}{h}\right)^2 x^2 \, dx = \pi \left(\frac{r^2}{h^2}\right) \int_0^h x^2 \, dx$$

$$= \pi \left(\frac{r^2}{\cancel{h^2}}\right) \cdot \frac{\cancel{h^3}\, h}{3} = \frac{1}{3} \pi r^2 \cdot h$$

Sample final exam

#1 The two parts of f are continuous.

We need to make sure two pieces match correctly.

From definition of f for $x \le 0$ we get

$$f(0) = 4e^0 + 0 - k = 4 - k.$$

To match with $x > 0$. we need $\lim_{x \to 0^+} f(x) = 4-k$.

$x > 0$ gives $f(x) = \frac{\sin kx}{x}$

$$\lim_{x \to 0^+} k \cdot \frac{\sin kx}{kx} = k \cdot 1 = k$$

So we need $k = 4 - k$

$2k = 4$

$k = 2$.

#2 From graph, determine

$$\lim_{x \to -2^-} \frac{|f(|x|) - 2|}{f(x) + 1} = \frac{1}{3}$$

As $x \to -2^-$, $f(x) \longrightarrow +2^-$ so $f(x) + 1 \longrightarrow 2 + 1 = 3$.

$|x| \longrightarrow +2^+$ so $f(|x|) \longrightarrow +3^-$

so $f(|x|) - 2 \longrightarrow 3 - 2 = 1$

$|f(|x|) - 2| \longrightarrow 1$