Suppose f is continuous on $a \le x \le b$

$$A(x) = \int_a^x f(t)\,dt.$$

$$A(a) = 0.$$

<u>FTC I</u> $A'(x) = f(x)$

<u>FTC II</u> $\int_a^b f(t)\,dt = A(b) - A(a)$

So average/mean value of f is

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{A(b)-A(a)}{b-a}$$ = secant slope of function A from $(a, A(a))$ to $(b, A(b))$.

$$= \text{by MVT } A'(c) \underset{\substack{\uparrow \\ \text{FTC I}}}{=} f(c)$$

This says the <u>average/mean value of f</u> equals some value $f(c)$ of at interior

Suppose f is integrable on interval $a \le x \le b$

$$\frac{1}{(b-a)} \int_a^b f(x)\,dx$$ is called average/mean value of f on the interval.

Examples ① If $T(t)$ is temperature $0 \le t \le 24$, then

$$\frac{1}{24} \int_0^{24} T(t)\,dt$$ average/mean temp over 24 hours.

② If $v(t)$ is velocity over $a \le t \le b$.

$$\int_a^b v(t)\,dt = \text{net displacement}$$

$$\frac{1}{b-a} \int_a^b v(t)\,dt =$$ is "average/mean" speed.

Substitution   Reverse of the chain rule

WW9 #12   Find $\displaystyle\int_0^1 \frac{dx}{\sqrt{x} + 4\sqrt[3]{x}}$ $\sqrt{\ }$ $\sqrt[3]{\ }$

Take $x = u^6$ so that $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$

$x = 0 \longleftrightarrow u = 0$

$x = 1 \longleftrightarrow u = 1$

$\frac{dx}{du} = 6u^5$ so $dx = 6u^5\,du$

$$\int_0^1 \frac{dx}{\sqrt{x} + 4\sqrt[3]{x}} = \int_0^1 \frac{6u^5\,du}{u^3 + 4u^2} = \int_0^1 \frac{6u^3\,du}{u+4}$$

$$\begin{array}{r|l} & u^2 - 4u + 16 \\ \hline u+4 & u^3 + 0u^2 + 0u + 0 \\ & u^3 + 4u^2 \\ \hline & -4u^2 \\ & -4u^2 - 16u \\ \hline & 16u \\ & 16u - 64 \\ \hline & 64 \end{array}$$

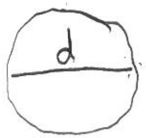
$$= 6\int_0^1 \left((u^2 - 4u + 16) + \frac{64}{u+4} \right) du$$

$$= 6\left\{ \frac{u^3}{3} - 2u^2 + 16u + 64\ln(u+4) \Big|_0^1 \right\}$$

$$= 6\left\{ \frac{1}{3} - 2 + 16 + 64\left(\ln(5) - \ln(4)\right) \right\}.$$

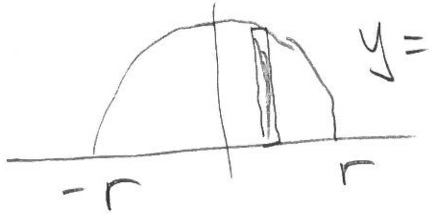
Area of circle

Number $\pi = 3.14159\ldots$



$$\pi = \frac{\text{circumference}}{\text{diameter}}$$

Area of circle of radius r is πr^2



$y = \sqrt{r^2 - x^2}$

half area circle is $\displaystyle\int_{-r}^{r} y\,dx = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$

Find definite integral by substitution $x = r\sin\theta$ $\quad dx = r\cos\theta\,d\theta$

$x = r \longleftrightarrow \sin\theta = 1$

$\theta = \frac{\pi}{2}$

$x = -1 \longleftrightarrow \theta = -\frac{\pi}{2}$

$$\sqrt{r^2 - x^2} = \sqrt{r^2 - r^2\sin^2\theta} = r\sqrt{1 - \sin^2\theta} = r\cos\theta$$

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta \; r\cos\theta\,d\theta = r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

To find $\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$, we use trig identify

$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ $\quad A = B = \theta$.

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

so $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

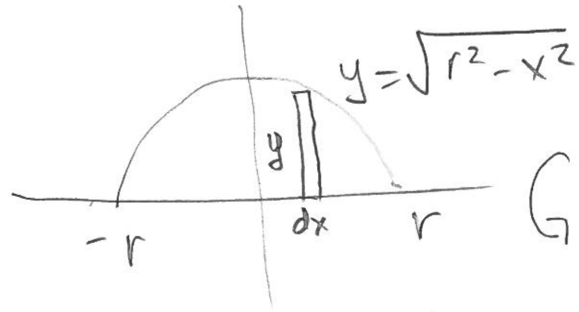
$$\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (1 - \cos 2\theta)\, d\theta = \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{2} - 0\right) - \left(-\frac{\pi}{2} - 0\right)\right) = \frac{\pi}{2}$$

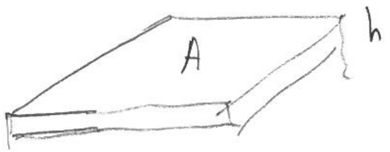
So half area circle $= r^2 \dfrac{\pi}{2}$

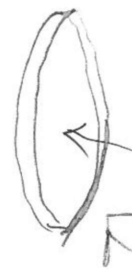
area circle $= \pi r^2$.

Volume of sphere radius r



rotate around x-axis





rectangle $y\,dx$ rotates to disk.

area of disk is πy^2

height is dx

$$\text{Volume of sphere} = \int_{-r}^{r} \pi y^2\,dx$$

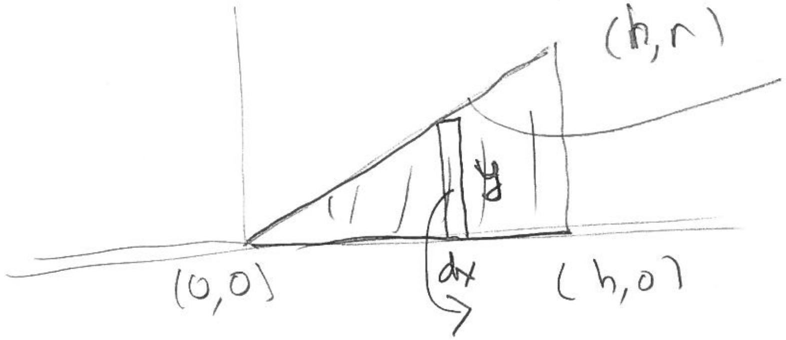
The volume $= \pi y^2 dx$

$$= \int_{-r}^{r} \pi (r^2 - x^2)\,dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2)\,dx = \pi \left\{ r^2 x - \frac{x^3}{3} \Big|_{-r}^{r} \right\}$$

$$= \pi \left\{ \left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{(-r)^3}{3}\right) \right\} = \frac{4\pi r^3}{3}.$$

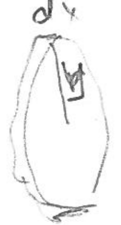
Volume of cone



$y = \frac{r}{h} x$

rotate  about x-axis to get cone height h, base radius r.

rectangle $y\,dx$ rotates to cylinder with height dx, radius y



$$\pi y^2 dx$$

$$\text{Volume of cone} = \int_0^h \pi y^2 dx = \int_0^h \pi \frac{r^2}{h^2} x^2 dx$$

$$= \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3}$$

$$= \frac{\pi r^2 \cdot h}{3} = \frac{1}{3} \cdot (\text{base area}) \cdot h.$$

Sample exam

\# 1 Determine k so that

$$f(x) = \begin{cases} 4e^{x} + x - k & x \leq 0 \\ \dfrac{\sin kx}{x} & x > 0 \end{cases}$$

is continuous.

Continuous means $\lim\limits_{x \to a} f(x) = f(a)$, f continuous on each piece. Choose k so pieces fit together to be continous.

$$f(0) = 4e^{0} + 0 - k = 4 - k.$$

So $\lim\limits_{x \to 0^+} \dfrac{\sin kx}{x}$ must be $4-k$.

$$\lim_{x \to 0^+} \frac{\sin kx}{x} = \lim_{x \to 0^+} k \cdot \frac{\sin kx}{kx} = k \cdot 1$$

Need $4-k=k$

$4=2k$

$k=2$.

#2 Find $\displaystyle \lim_{x \to -2^-} \frac{|f(|x|) - 2|}{f(x) + 1}$ $\to 1$

As $x \to -2^-$ we see $f(x) \to 2^-$ so $\dfrac{1}{f(x)+1} \to \dfrac{1}{3}$.

As $x \to -2^-$ we have $|x| \to 2^+$

$$f(|x|) \to 3$$

$$f(|x|) - 2 \to 3 - 2 = 1$$

$$|f(|x|) - 2| \to 1$$

So $\displaystyle \lim_{x \to -2^-} \frac{|f(|x|) - 2|}{f(x) + 1} = \frac{1}{3}$.