Sample Final

#23 Based on graph determine

(a) $\lim\limits_{x \to 2^-} \dfrac{f(x)^2 + 3f(x) + 2}{f(x)^2 - 1}$.

As $x \to 2^-$, the graph shows $f(x) \to -1^+$

So $f(x)^2 + 3f(x) + 2 \to (-1)^2 + 3(-1) + 2 = 0$

and $f(x)^2 - 1 \longrightarrow (-1)^2 - 1 = 0$

The limit is indeterminant. $\frac{0}{0}$.

We factor: $(f(x)^2 + 3f(x) + 2) = (f(x)+1)(f(x)+2)$

$f(x)^2 - 1 = (f(x)+1)(f(x)-1)$

so $\dfrac{f(x)^2 + 3f(x) + 2}{f(x)^2 - 1} = \dfrac{\cancel{(f(x)+1)}(f(x)+2)}{\cancel{(f(x)+1)}(f(x)-1)} = \dfrac{f(x)+2}{f(x)-1}$ and $\begin{matrix} f(x)+2 \to -1+2 = 1 \\ f(x)-1 \to -1-1 = -2 \end{matrix}$

so $\lim\limits_{x \to 2^-} (\ ) = \dfrac{1}{-2} = -\dfrac{1}{2}$.

(b) If we restrict domain to $-6 < x < -2$, the function is one-to-one. (function decreasing with range of values $-2 < f(x) < 4$).

Find $\lim_{x \to -2} f^{-1}(x) = -2$

(c) Find $\lim_{x \to 0} \dfrac{\int_0^{x^2} (t-1) f(t)\,dt}{4x^2}$

$\int_0^{0^2} (t-1) f(t)\,dt = 0$

$4x^2 \big|_{x=0} = 0$

indeterminant $\frac{0}{0}$ limit.

Use L'hopital's rule

$\lim_{x \to 0} \dfrac{(x^2-1) f(x^2) \cancel{2x}}{\cancel{8x}\ 4}$

$(4x^2)' = 4 \cdot 2x$

$\left(\int_0^{x^2} (t-1) f(t)\,dt \right)' = \left(A(x^2) \right)' = A'(x^2) \cdot 2x$

$A(x) =$ area function $\int_0^x (t-1) f(t)\,dt$

FTC $A'(z) = (z-1) f(z)$

So $\lim_{x \to 0} \dfrac{(x^2-1) f(x^2)}{4} = \dfrac{(0^2-1) \cdot 4}{4} = -1$.

$(\lim_{x \to 0} f(x^2) = 4)$

#24 $f(x) = x^5 - 10x^3 + 500$

(a) Determine intervals of increase / decrease.

$f' > 0$ $f' < 0$

$f'(x) = 5x^4 - 30x^2 + 0 = 5x^2(x^2 - 6)$

$f' > 0$		$f' < 0$		$f' < 0$		$f' > 0$
	$-\sqrt{6}$		0		$\sqrt{6}$	

increase is $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty)$

decrease is $(-\sqrt{6}, 0) \cup (0, \sqrt{6})$ "OR" $(-\sqrt{6}, \sqrt{6})$

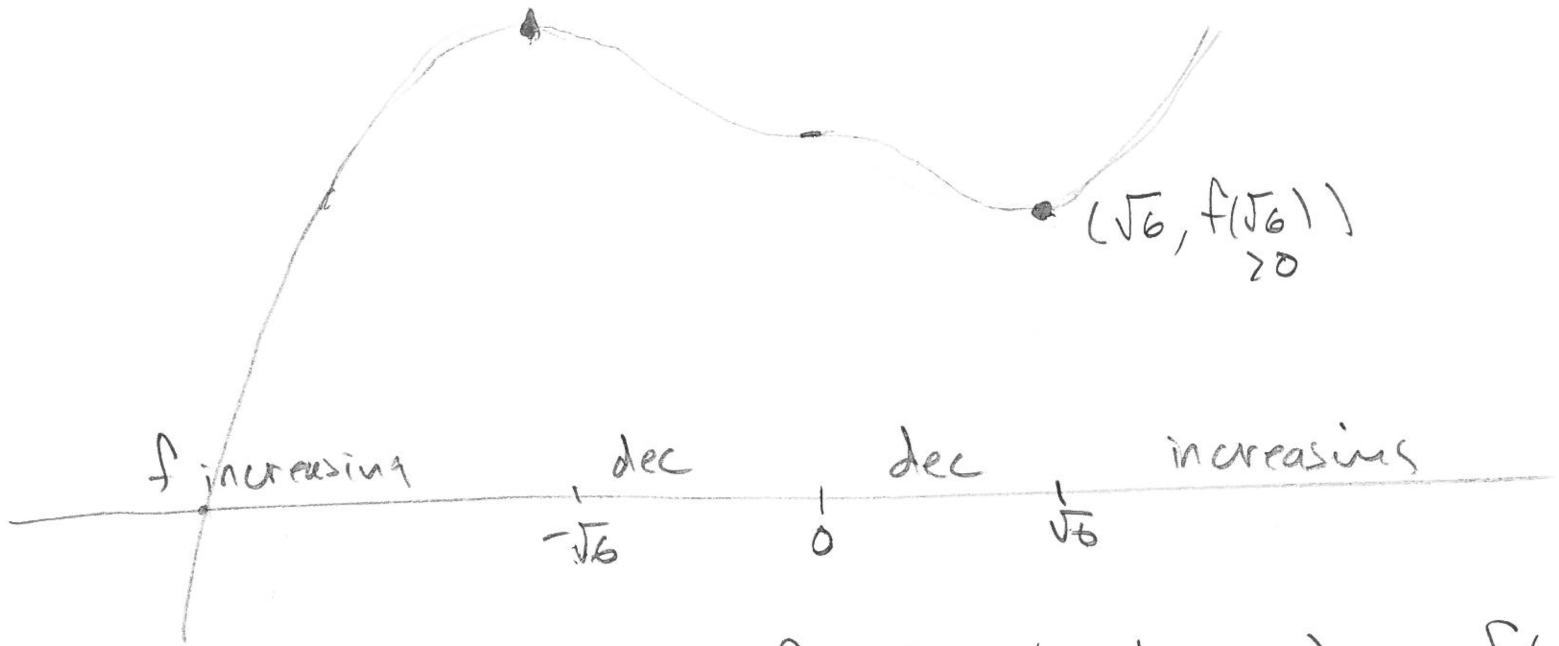
(b) Determine interval where concave up $\longleftrightarrow$ $f'' > 0$

$f''(x) = 20x^3 - 60x = 20x(x^2 - 3)$

$f'' < 0$		$f'' > 0$		$f'' < 0$		$f'' > 0$
	$-\sqrt{3}$		0		$\sqrt{3}$	

concave up when $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

(c) Determine how many roots $f(x) = x^5 - 10x^3 + 500$ has



Important places to check are $f(-\sqrt{6})$ (local max) $f(-\infty)$
$f(\sqrt{6})$ (local min) $f(+\infty)$

x	$f(x)$
$\sqrt{6}$	$(\sqrt{6})^5 - 10(\sqrt{6})^3 + 500$

$$(\sqrt{6})^5 - 10(\sqrt{6})^3 + 500 = 6\cdot 6\sqrt{6} - 10\cdot 6\sqrt{6} + 500 = 500 - 24\sqrt{6}$$
$$36\sqrt{6} - 60\sqrt{6} + 500 \qquad > 0$$

Since $f(\sqrt{6}) > 0$, then no root on $(\sqrt{6}, \infty)$ because increasing

Also f decreasing on $(-\sqrt{6}, \sqrt{6})$ so $f > f(\sqrt{6}) > 0$ on this interval

$\Rightarrow f(-\sqrt{6}) > 0$. Since $f(-\infty) = -\infty$, we see f crosses x axis exactly once in interval $(-\infty, -\sqrt{6})$. 1 ROOT

25 $f(x) = \dfrac{3 - x^2 + 2x^3}{x^3} = \dfrac{3}{x^3} - \dfrac{1}{x} + 2$

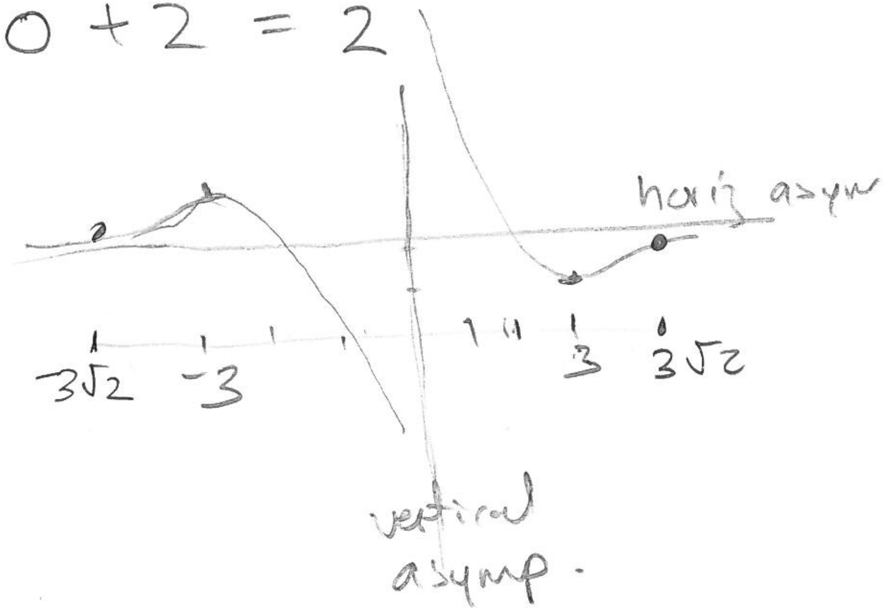
odd function

(a) Vertical asymptote at $x = 0$

Since $\lim_{x \to +\infty} \left(\frac{3}{x^3} - \frac{1}{x} + 2 \right) = 0 - 0 + 2 = 2$

Similarly $\lim_{x \to -\infty} \left(\frac{3}{x^2} - \frac{1}{x} + 2 \right) = 2$

So $y = 2$ horizontal asymptote.



Graph of $\frac{3}{x^3} - \frac{1}{x}$

(b) Where is $f'(x) = 0$:

$f'(x) = \left(\frac{3}{x^3} - \frac{1}{x} + 2 \right)' = 3(-3)x^{-4} - (-1)x^{-2} + 0 = \dfrac{x^2 - 9}{x^4}$

where $f' = 0$ is where $x^2 - 9 = 0$ so $x = \pm 3$.

(c) find inflection points (where f'' changes sign) 6

$$f''(x) = 3(-3)(-4)x^{-5} - (-1)(-2)x^{-3}$$

$$= \frac{36-2x^2}{x^5} = \frac{2(18-x^2)}{x^5}.$$

$f''=0$ when $18-x^2=0$ so $x = \pm\sqrt{18} = \pm 3\sqrt{2}$

and f'' actually changes sign at $3\sqrt{2}$ and $-3\sqrt{2}$

So these are inflection points

$$3\sqrt{2} = 3 \cdot 1.41421\ldots$$

local min at $x=3$, $f(3) = \frac{3}{3^3} - \frac{1}{3} + 2 = \frac{1}{9} - \frac{1}{3} + 2 = -\frac{2}{9} + 2$

local max at $x=-3$ $f(-3) = \left(\frac{3}{3^3} + \frac{1}{3}\right) + 2 = \frac{2}{9} + 2.$

inflection pt at $x=\sqrt{18}$ $f(\sqrt{18}) = \frac{3}{(\sqrt{18})^3} - \frac{1}{\sqrt{18}} + 2$

$$f(-\sqrt{18}) = -\left(\frac{3}{\sqrt{18}^3} - \frac{1}{\sqrt{18}}\right) + 2$$

#26 Based on table find

(a) (i) $\int_0^2 \left(2f'(x) - 6xg'(x^2)\right) dx$

Use FTC II by finding antiderivative of $2f'(x) - 6xg'(x^2)$

$\left(2f(x) - 3g(x^2)\right)' = 2f'(x) - 3g'(x^2)\cdot 2x$ $\quad 3g'(x^2)\cdot 2x$

so $\int_0^2 \left(2f'(x) - 6xg'(x)\right) dx = \left(2f(x) - 3g(x^2)\right)\Big|_0^2$

$= 2\left(f(2) - f(0)\right) - 3\left(g(4) - g(0)\right)$

$= 2(0-2) - 3(2-3)$

$= -4+3 = -1$.

(ii) $\int_{-1}^{1} \left(f(x+3)\right)^4 f'(x+3)\, dx$

has antiderivative $\dfrac{\left(f(x+3)\right)^5}{5}$ Then use FTC II.

(b) Does the graph $y = 2f(x) - g(x) + h(x)$ "must" have a horizontal tangent line?

$\Leftrightarrow \frac{dy}{dx} = 2f'(x) - g'(x) + h'(x) = 0$ somewhere.

Based on table $\left.\frac{dy}{dx}\right|_{x=0} = \ldots = 5$

$\left.\frac{dy}{dx}\right|_{x=2} = \ldots = 7$

$\left.\frac{dy}{dx}\right|_{x=4} = \quad = 12$

Since ALL positive, we cannot conclude $\frac{dy}{dx} = 0$ somewhere