Sample Final

#23 From graph find

(a) $\displaystyle\lim_{x \to 2^-} \frac{f(x)^2 + 3f(x) + 2}{f(x)^2 - 1}$

As $x \to 2^-$, we have $f(x) \to -1$

so $f(x)^2 + 3f(x) + 2 \to (-1)^2 + 3(-1) + 2$
$= 0$

and $f(x)^2 - 1 \to (-1)^2 - 1 = 0$

indeterminant limit. $\frac{0}{0}$.

$f(x)^2 - 1 = (f(x) - 1)(f(x) + 1)$

$f(x)^2 + 3f(x) + 2 = (f(x) + 1)(f(x) + 2)$

so $$\frac{f(x)^2 + 3f(x) + 2}{f(x)^2 - 1} = \frac{(f(x)+2)\cancel{(f(x)+1)}}{\cancel{(f(x)+1)}(f(x)-1)} = \frac{f(x)+2}{f(x)-1} \to \frac{-1+2}{-1-1} = \frac{1}{-2} = -\frac{1}{2}$$

(b) f restricted domain $-6 < x < -2$ is one-to-one

(so there is inverse)

Find $\displaystyle\lim_{x \to -2} f^{-1}(x)$

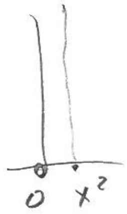
From graph $\displaystyle\lim_{x \to -2} f(x) = -2^{+}$ | Also range of f on $-6 < x < -2$ is $-2 < y < 4$

As $y \to -2^{+}$, we see $f^{-1}(y) \to -2^{-}$, so $\displaystyle\lim_{y \to -2} f^{-1}(y) = -2$.

(c) $\displaystyle\lim_{x \to 0} \frac{\int_0^{x^2} (t-1) f(t)\, dt}{4x^2} = -1$

$\displaystyle\lim_{x \to 0} \int_0^{x^2} (t-1) f(t)\, dt = 0$

$A(x^2)$



$\displaystyle\lim_{x \to 0} 4x^2 = 0$

indeterminant $\frac{0}{0}$

Since indeterminant, can try L'hopital.

$$\frac{A'(x^2) \cdot 2x}{4 \cdot 2x} = \frac{(x^2-1) f(x^2)\, \cancel{2x}}{4 \cdot \cancel{2x}} = \frac{(x^2-1) f(x^2)}{4} \xrightarrow[x \to 0]{} \frac{(0^2-1) f(0)}{4} = \frac{-4}{4}$$

$= -1$

24 $f(x) = x^5 - 10x^3 + 500$

(a) Find intervals of increase decrease

$f' > 0$ $f' < 0$

$f'(x) = 5x^4 - 10 \cdot 3x^2 + 0 = 5x^2(x^2 - 6)$

$f' > 0$ $f' < 0$ $f' < 0$ $f' > 0$

$-\sqrt{6}$ 0 $\sqrt{6}$

intervals of increase $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \infty)$

intervals of decrease $(-\sqrt{6}, 0) \cup (0, \sqrt{6})$.

(b) Where is f concave up?

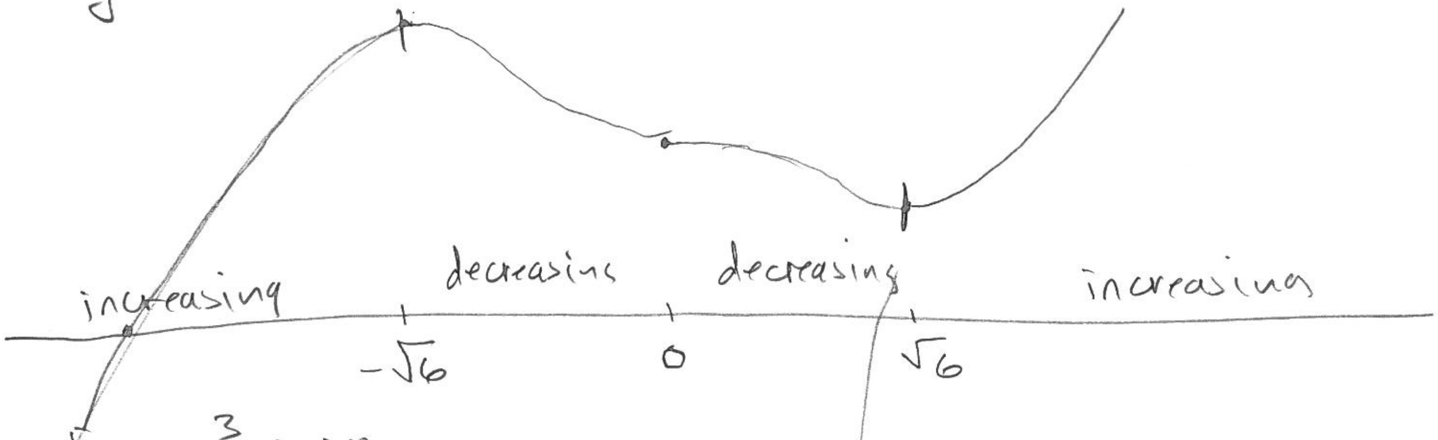
$f'' = 5 \cdot 4 \cdot x^3 - 10 \cdot 3 \cdot 2x = 20 \cdot x(x^2 - 3)$

$f'' < 0$ $f'' > 0$ $f'' < 0$ $f'' > 0$

$-\sqrt{3}$ 0 $\sqrt{3}$

Concave up when $f'' > 0$ so $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

(c) How many real roots does f have?



x	$f(x) = x^5 - 10x^3 + 500$
"$-\infty$"	"$-\infty$"
$-\sqrt{6}$	$6 \cdot 6(-\sqrt{6}) - 10 \cdot 6 \cdot (-\sqrt{6}) + 500 > 0$
0	500
$\sqrt{6}$	$6 \cdot 6\sqrt{6} - 10 \cdot 6(\sqrt{6}) + 500$ > 0

$(-\sqrt{6})^2 = 6$

$\cancel{96}(-\sqrt{6}) + 500$

$-24(-\sqrt{6}) + 500$

Conclude by Intermediate Value Theorem f has 1 root.

#25 $f(x) = \dfrac{3 - x^2 + 2x^3}{x^3} = 3x^{-3} - x^{-1} + 2$

(a) vertical asymptote. Need a so that $\lim_{x \to a} f(x) = \pm\infty$.

where $a^3 = 0$ so $a = 0$.

vertical line $x=0$ is vertical asymptote.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 3x^{-3} - x^{-1} + 2 = \frac{3}{\infty} - \frac{1}{\infty} + 2 = 0 - 0 + 2 = 2$

similarly for $x \to -\infty$. So $y=2$ is horizontal asymptote.

(b) Find where $f'(x) = 0$.

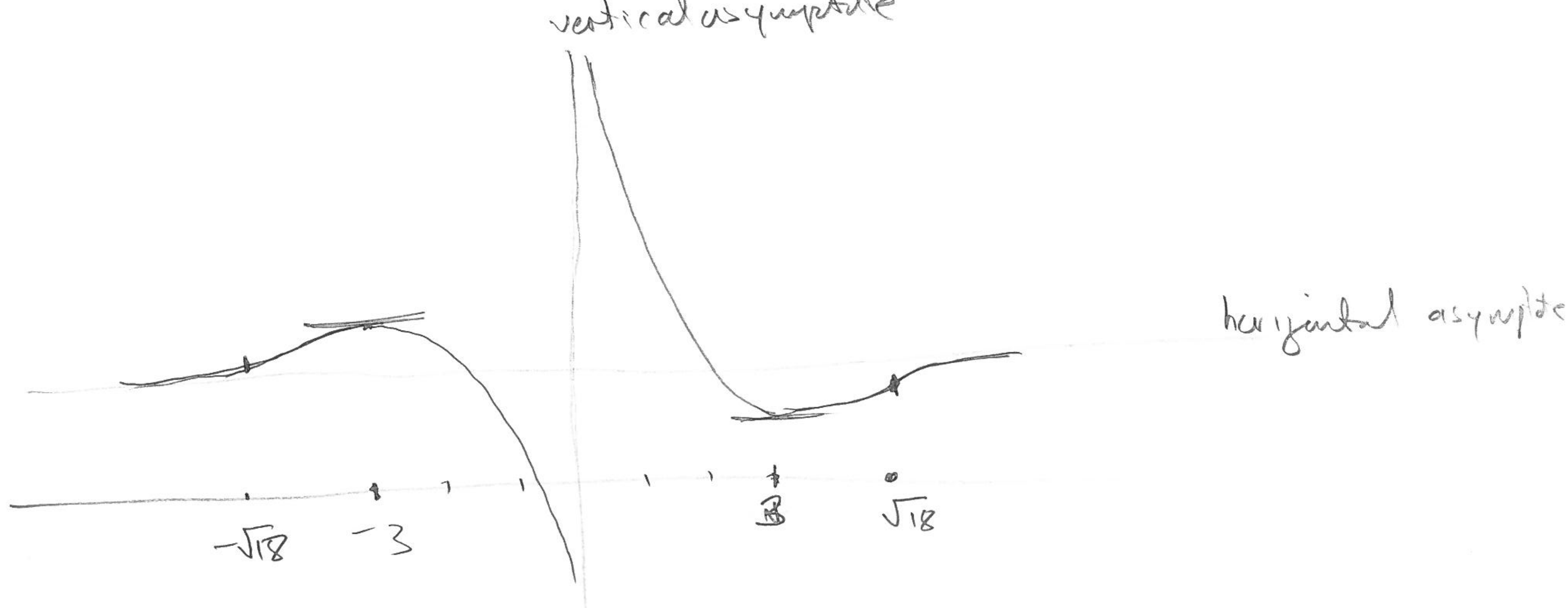
$f'(x) = 3(-3)x^{-4} - (-1)x^{-2} + 0 = \frac{-9}{x^4} + \frac{1}{x^2}\frac{x^2}{x^2} = \frac{x^2 - 9}{x^4}$.

Need $x^2 - 9 = 0$ so $x = \pm 3$.

(c) Find inflection points. Where f'' changes sign.

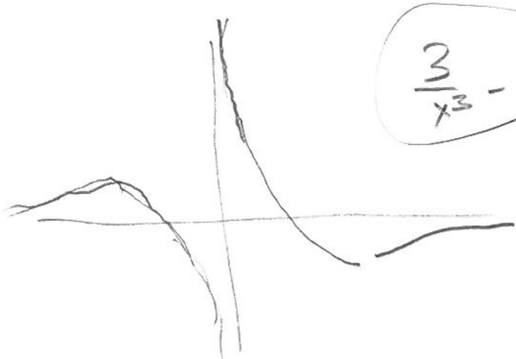
$f''(x) = 3(-3)(-4)x^{-5} - (-1)(-2)x^{-3} = \dfrac{36 - 2x^2}{x^5} = \dfrac{2(18 - x^2)}{x^5}$

f'' changes sign at $\pm\sqrt{18} = \pm 3\sqrt{2} \doteq \pm 3 \cdot (1.4141)$



Graph function $f(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right) + 2$

odd function



$\frac{3}{x^3} - \frac{1}{x}$ has graph

#26

(a)(i) Evaluate definite integral $\int_0^2 (2f'(x) - 6x\,g'(x^2))\,dx$

By FTC II we should seek antiderivative of $2f'(x) - 6x\,g'(x^2)$

$2f(x) - 3g(x^2)$ has derivative $2f'(x) - 3g'(x) \cdot 2x$

So FTC II $\int_0^2 (2f'(x) - 6x\,g'(x^2))\,dx = 2f(x) - 3g(x^2) \Big|_0^2$

$= 2(f(2) - f(0)) - 3(g(4) - g(0))$

$= 2(0 - 2) - 3(2 - (-2))$

$= -4 - 12 = -16.$

(iii) Evaluate $\int_{-1}^{1} f(x+3)^4 \, f'(x+3) \, dx$

Need antiderivative of $f(x+3)^4 \, f'(x+3)$

$\dfrac{f(x+3)^5}{5}$ is antiderivative of $\dfrac{5 f(x+3)^4}{5} \cdot f'(x+3) \cdot (1+0)$

So FTC II

$$\int_{-1}^{1} f(x+3)^4 f'(x+3) \, dx = \left. \frac{f(x+3)^5}{5} \right|_{-1}^{1}$$

$$= \frac{f(4)^5}{5} - \frac{f(2)^5}{5} = \frac{1^5}{5} - \frac{0^5}{5} = \frac{1}{5}$$

(b) Can $y = 2f(x) - g(x) + h(x)$ have horizintal tangent line?

$$\frac{dy}{dx} = 2f'(x) - g'(x) + h'(x)$$

Based on table of values is $\frac{dy}{dx} = 0$?

	$\frac{dy}{dx}$
0	$2f'(0) - g'(0) + h'(0) = 2\cdot 0 - (-2) + 3 = 5$
2	$2f'(2) - g'(2) + h'(2) = 2\cdot 3 - 0 + 1 = 7$
4	$2f'(4) - g'(4) + h'(4) = 2\cdot 6 - 2 + 2 = 12$

Since $5, 7, 12 > 0$, it is not nessarily true that $\frac{dy}{dx} = 0$ somewhere in interval $\underline{0 \le x \le 4}$. (False)