

FINAL EXAMINATION SOLUTIONS - MATH 2121, FALL 2018.

Name:

Student ID:

Email:

Tutorial:    T1A        T1B        T2A        T2B        T3A        T3B

Problem #	Max points possible	Actual score
1	20	
2	15	
3	10	
4	15	
5	20	
6	10	
7	15	
8	15	
Total	120	

You have **180 minutes** to complete this exam.

**No books, notes, or electronic devices can be used on the test.**

Clearly label your answers by putting them in a  box.

Partial credit can be given on some problems if you show your work. Good luck!

**Problem 1.** (4 + 4 + 4 + 4 + 4 = 20 points)

- (a) State the definition of a
- linear transformation*
- $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$
- .

A *linear transformation*  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a function such that

$$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v)$$

for all vectors  $u, v \in \mathbb{R}^n$  and scalars  $c \in \mathbb{R}$ .

- (b) Suppose
- $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$
- ,
- $v_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$
- ,
- $v_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$
- , and
- $w = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$
- .

Determine if  $w$  is in the span of  $v_1, v_2$ , and  $v_3$ .

Justify your answer to receive full credit.

The reduced echelon form of  $[v_1 \ v_2 \ v_3 \ w]$  is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2/9 \\ 0 & 0 & 1 & -9/7 \end{bmatrix}.$$

Since there is no pivot in the last column, we conclude that  $w$  is in the span of  $v_1, v_2, v_3$ , and more specifically that

$$w = (-2/9)v_2 + (-9/7)v_3.$$

- (c) State the definition of the
- dimension*
- a subspace
- $V$
- of
- $\mathbb{R}^n$
- .

The dimension of a subspace  $V$  is the size of any basis for  $V$ .

(d) Let  $v_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$ ,  $v_3 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ -2 \end{bmatrix}$ ,  $v_4 = \begin{bmatrix} 9 \\ 8 \\ 4 \\ 2 \end{bmatrix}$ , and  $v_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ .

Determine if the vectors  $v_1, v_2, v_3, v_4, v_5$  are linearly independent.

Justify your answer to receive full credit.

The vectors are not linearly independent, since they consist of 5 vectors in  $\mathbb{R}^4$ , but  $n$  vectors in  $\mathbb{R}^m$  are never linearly independent if  $n > m$ .

(e) Consider the matrix

$$A = \begin{bmatrix} 1 & 7 & 0 \\ -2 & -3 & -3 \end{bmatrix}.$$

Find a  $3 \times 2$  matrix  $B$  such that

$$Au \bullet v = u \bullet Bv$$

for all  $u \in \mathbb{R}^3$  and  $v \in \mathbb{R}^2$ , where  $\bullet$  denotes the vector inner product.

Since  $Au \bullet v = (Au)^T v = u^T A^T v$  and  $u \bullet Bv = u^T Bv$ , the desired matrix is

$$B = A^T = \begin{bmatrix} 1 & -2 \\ 7 & -3 \\ 0 & -3 \end{bmatrix}.$$

**Problem 2.** (15 points) In the following statements,  $A, B, C$ , etc., are matrices (with all real entries), and  $b, u, v, w, x$ , etc., are vectors, unless otherwise noted.

Indicate which of the following is TRUE or FALSE.

One point will be given for each correct answer (no penalty for incorrect answers).

- (1) Every linear system with fewer equations than variables has a solution.

FALSE      Consider the system 
$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

- (2) If  $w$  is a linear combination of  $u$  and  $v$  in  $\mathbb{R}^n$ , then  $u$  is a linear combination of  $v$  and  $w$ .

FALSE      Suppose  $w = 0$  but  $u$  and  $v$  are linearly independent.

- (3) If  $A$  is an  $n \times n$  matrix and  $I$  is the  $n \times n$  identity matrix and  $A^m = 0$  for some positive integer  $m$ , then  $I - A$  is invertible.

TRUE      Check that  $(I - A)^{-1} = I + A + A^2 + \cdots + A^{m-1}$ .

- (4) If  $A$  is a  $2 \times 2$  matrix and  $\det A = 0$ , then one row of  $A$  is a scalar multiple of the other row.

TRUE

Since  $\det A^T = \det A = 0$ , the columns of  $A^T$  are linearly dependent.

- (5) If  $A$  and  $B$  are row equivalent  $m \times n$  matrices and the columns of  $A$  span  $\mathbb{R}^m$ , then so do the columns of  $B$ .

TRUE

The columns of  $A$  span  $\mathbb{R}^m$  if  $A$  has a pivot in every row. If  $A$  has this property and  $B$  is row equivalent to  $A$ , then  $B$  also has a pivot in every row so its columns span  $\mathbb{R}^m$ .

- (6) If  $A$  is an  $m \times n$  matrix and the linear system  $Ax = b$  has more free variables than basic variables, then  $\text{rank } A < \frac{n}{2}$ .

TRUE

The rank of  $A$  is the number of basic variables, and the sum of the number of free and basic variables is  $n$ .

- (7) If  $A$  is an  $m \times n$  matrix then the function  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  defined by  $T(x) = Ax$  for  $x \in \mathbb{R}^n$  is one-to-one only if  $\text{Nul}A = \{0\}$ .

TRUE

Assume  $\text{Nul}A = \{0\}$ . If  $x, y \in \mathbb{R}^n$  and  $x \neq y$ , then  $x - y \neq 0$ , so

$$0 \neq A(x - y) = Ax - Ay = T(x) - T(y)$$

and  $T(x) \neq T(y)$ . Therefore  $T$  is one-to-one.

- (8) If  $A$  is a  $3 \times 3$  matrix and at least 6 entries in  $A$  are zero, then  $A$  is not invertible.

FALSE Consider  $A = I$  the  $3 \times 3$  identity matrix

- (9) Each eigenvector of an invertible square matrix  $A$  is an eigenvector of  $A^{-1}$ .

TRUE Since if  $Av = \lambda v$  then  $A^{-1}v = \lambda^{-1}v$ .

- (10) If  $A$  is an  $n \times n$  matrix with fewer than  $n$  distinct eigenvalues, then  $A$  is not diagonalizable.

FALSE

The identity matrix  $A = I$  has only one distinct eigenvalue but is diagonal and there diagonalizable.

- (11) An  $n \times n$  matrix can have  $n$  distinct eigenvalues and exactly  $n - 1$  real eigenvalues.

FALSE

In the problem instructions, we are told that  $A$  has all real entries. Therefore if  $\lambda$  is a complex eigenvalue that is not real then  $\bar{\lambda}$  is a second complex eigenvalue that is not real.

- (12) If the columns of  $A$  are orthonormal then  $A$  has at least as many rows as columns.

TRUE

If the columns are orthonormal then they are linearly independent.

- (13) If  $W$  is a subspace of  $\mathbb{R}^n$  then  $(W^\perp)^\perp = W$ .

TRUE

We have  $W \subset (W^\perp)^\perp$  since  $u \bullet v = v \bullet u$ . The dimension of  $W^\perp$  is  $n - \dim W$  and the dimension of  $(W^\perp)^\perp = n - \dim W^\perp = \dim W$ . Since  $W$  is a subspace of  $(W^\perp)^\perp$  with the same dimension,  $(W^\perp)^\perp = W$ .

- (14) If  $A$  is symmetric then  $A$  has all real eigenvalues.

TRUE      This was a theorem presented in class.

- (15) Matrices with only real entries that have non-real eigenvalues can have singular value decompositions involving matrices with only real entries.

TRUE

Every matrix has an SVD and every SVD only involves real matrices

**Problem 3.** (5 + 5 = 10 points)

(a) Compute the inverse of the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

One checks that

$$\text{RREF} \left( \begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}$$

so the inverse matrix is

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}.$$

(b) Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}.$$

Performing several row operations gives

$$\begin{aligned} \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 6 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & 6 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & -13 & -13 \\ 0 & 0 & 6 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & -13 & -13 \\ 0 & 0 & 0 & -5 \end{bmatrix}. \end{aligned}$$

None of these row operations swapped or rescaled rows, so the determinant of the last matrix is equal to  $\det A$ . The last matrix is triangular so its determinant is  $1 \cdot 1 \cdot -13 \cdot -5 = 65$ . Therefore

$$\boxed{\det A = 65.}$$



**Problem 4.** (5 + 5 + 5 = 15 points) Let

$$\mathcal{V} = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} : a, b, c, d, e, f, g, h, i \in \mathbb{R} \right\}$$

be the vector space of  $3 \times 3$  matrices.

Define  $L : \mathcal{V} \rightarrow \mathcal{V}$  to be the linear transformation

$$L \left( \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = \begin{bmatrix} a+e+i & b+f & c \\ d+h & a+e+i & b+f \\ g & d+h & a+e+i \end{bmatrix}.$$

(a) Find a basis for the subspace  $\mathcal{R} = \{L(A) : A \in \mathcal{V}\}$ . What is  $\dim \mathcal{R}$ ?

The matrices

$$\left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right]$$

form a basis for  $\mathcal{R}$ , so  $\dim \mathcal{R} = 5$ .

(b) Find a basis for the subspace  $\mathcal{N} = \{A \in \mathcal{V} : L(A) = 0\}$ . What is  $\dim \mathcal{N}$ ?

We have  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \mathcal{N}$  if and only if

$$c = b + f = a + e + i = d + h = g = 0$$

in which case we can write

$$A = \begin{bmatrix} a & b & 0 \\ d & e & -b \\ 0 & -d & -a - e \end{bmatrix}.$$

The matrices

$$\left[ \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right]$$

therefore form a basis for  $\mathcal{N}$ , and  $\dim \mathcal{N} = 4$ .

Note that  $\dim \mathcal{R} + \dim \mathcal{N} = 9 = \dim \mathcal{V}$  which is consistent with the rank-nullity theorem.

- (c) A number  $\lambda \in \mathbb{R}$  is an *eigenvalue* for  $L$  if there exists a nonzero matrix  $A \in \mathcal{V}$  with  $L(A) = \lambda A$ , in which case say that  $A$  is an *eigenvector* for  $L$ .

Find the distinct eigenvalues  $\lambda$  for  $L$ . For each eigenvalue  $\lambda$ , provide a nonzero matrix  $A \in \mathcal{V}$  with  $L(A) = \lambda A$ .

Part (b) shows that  $\lambda = 0$  is an eigenvalue, with eigenvector  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ .

If  $\lambda \neq 0$  is an eigenvalue and  $L(A) = \lambda A$ , then  $A$  must have the form

$$A = \begin{bmatrix} a & b & c \\ d & a & b \\ e & d & a \end{bmatrix}$$

for some  $a, b, c, d, e$ . But then

$$L(A) = \begin{bmatrix} 3a & 2b & c \\ 2d & 3a & 2b \\ e & 2d & 3a \end{bmatrix} = \lambda A$$

so  $\lambda \in \{1, 2, 3\}$ .

An eigenvector for  $\lambda = 1$  is  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ .

An eigenvector for  $\lambda = 2$  is  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

An eigenvector for  $\lambda = 3$  is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ .

**Problem 5.** (5 + 10 + 5 = 20 points)

(a) Compute the distinct eigenvalues of the matrix  $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$ .

We have

$$\begin{aligned} \det(A - xI) &= \det \begin{bmatrix} 2-x & 0 & 0 \\ -3 & -1-x & -2 \\ 3 & 3 & 4-x \end{bmatrix} \\ &= (2-x)((-1-x)(4-x) - (-2)(3)) \\ &= (2-x)((x+1)(x-4) + 6) \\ &= (2-x)(x^2 - 4x + x - 4 + 6) \\ &= (2-x)(x^2 - 3x + 2) \\ &= (2-x)(2-x)(1-x). \end{aligned}$$

Therefore the distinct eigenvalues of  $A$  are 1 and 2.

(b) Again let  $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$ .

For each eigenvalue  $\lambda$  of  $A$ , find a basis for the eigenspace  $\text{Nul}(A - \lambda I)$ .

First suppose  $\lambda = 2$ . To find a basis of  $\text{Nul}(A - 2I)$  we row reduce

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -3 & -3 & -2 \\ 3 & 3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A - 2I).$$

This tells us that  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Nul}(A - 2I)$  if and only if

$$x = \begin{bmatrix} -x_2 - 2/3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for  $\text{Nul}(A - 2I)$  is given by the two vectors

$$\left[ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 0 \\ 1 \end{bmatrix} \right].$$

Next suppose  $\lambda = 1$ . To find a basis of  $\text{Nul}(A - I)$  we row reduce

$$A - I = \begin{bmatrix} 1 & 0 & 0 \\ -3 & -2 & -2 \\ 3 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A - I).$$

This tells us that  $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \text{Nul}(A - I)$  if and only if

$$x = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

so a basis for  $\text{Nul}(A - I)$  is given by the since vectors

$$\left[ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right].$$

(c) Continue to let  $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$ .

Determine if  $A$  is diagonalizable. If  $A$  is diagonalizable, then give an invertible matrix  $P$  and a diagonal matrix  $D$  such that

$$A = PDP^{-1}.$$

If  $A$  is not diagonalizable, give an explanation why.

The dimension the 2-eigenspace of  $A$  is 2 and the dimension of the 1-eigenspace of  $A$  is 1. Since these dimensions sum to 3 which is the number of columns of  $A$ , it follows that  $A$  is diagonalizable with  $A = PDP^{-1}$  for

$$P = \begin{bmatrix} -1 & -2/3 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Problem 6.** (5 + 5 = 10 points) Consider the symmetric matrix

$$A = \begin{bmatrix} 9/2 & 11/2 \\ 11/2 & 9/2 \end{bmatrix}.$$

(a) Find a  $2 \times 2$  orthogonal matrix  $U$  and a  $2 \times 2$  diagonal matrix  $D$  such that

$$A = UDU^T = UDU^{-1}.$$

The sum of the eigenvalues of  $A$  is  $\text{tr}A = 9$  while the product of the eigenvalues is  $\det A = 81/4 - 121/4 = -10$ . From these constraints, we can guess that the eigenvalues must be 10 and  $-1$ . (Alternatively, one can find the eigenvalues by factoring  $\det(A - xI)$ .)

To find eigenvalues corresponding to these eigenvalues, we row reduce

$$A + I = \begin{bmatrix} 11/2 & 11/2 \\ 11/2 & 11/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{RREF}(A + I)$$

which implies that  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$  is eigenvector with eigenvalue 1, while

$$A - 10I = \begin{bmatrix} -11/2 & 11/2 \\ 11/2 & -11/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{RREF}(A - 10I)$$

while implies that  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is an eigenvector with eigenvalue 10. These eigenvalues are already orthogonal, so converting them to unit vectors gives an orthonormal basis of  $\mathbb{R}^2$  consisting of the eigenvectors

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We conclude that  $A = UDU^T$  for the matrices

$$\boxed{U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 10 \end{bmatrix}.$$

(b) Continue to let

$$A = \begin{bmatrix} 9/2 & 11/2 \\ 11/2 & 9/2 \end{bmatrix}.$$

Find exact formulas for the functions  $a(n), b(n), c(n), d(n)$  such that

$$A^n = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}$$

for all positive integers  $n$ .

We have  $A^n = UD^nU^T$  where  $U$  and  $D$  as in the previous part. Therefore

$$\begin{aligned} A^n &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 10^n \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 \\ 0 & 10^n \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -(-1)^n & 10^n \\ (-1)^n & 10^n \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (-1)^n + 10^n & -(-1)^n + 10^n \\ -(-1)^n + 10^n & (-1)^n + 10^n \end{bmatrix}. \end{aligned}$$

Thus we have

$$A^n = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} = \begin{bmatrix} (10^n + (-1)^n)/2 & (10^n - (-1)^n)/2 \\ (10^n - (-1)^n)/2 & (10^n + (-1)^n)/2 \end{bmatrix}.$$

**Problem 7.** (5 + 5 + 5 = 15 points) Let  $A$  be the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

(a) Find an orthogonal basis for the column space of  $A$ .

By row reducing  $A$  to echelon form, one finds that the three columns of  $A$  are linearly independent and therefore give a basis for the column space. The first two columns are already orthogonal, while

$$\frac{\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}} = \frac{2+1}{1+1+1} = 1$$

and

$$\frac{\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}} = \frac{2+2-1}{1+1+1} = 1$$

The vector

$$\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

is therefore orthogonal to the first two columns and the three vectors

$$\left[ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right]$$

are an orthogonal basis for the column space of  $A$ .



(b) Find a least-squares solution to the linear system  $Ax = b$  where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

A least-squares solution is an exact solution to  $A^T Ax = A^T b$ , and we have

$$A^T A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 3 \\ 3 & 3 & 9 \end{bmatrix} \quad \text{and} \quad A^T b = \begin{bmatrix} 1 \\ 14 \\ 10 \end{bmatrix}.$$

Row reducing gives

$$\begin{aligned} [A^T A \quad A^T b] &= \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 3 & 3 & 14 \\ 3 & 3 & 9 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 3 & 3 & 14 \\ 0 & 3 & 6 & 9 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 3 & 3 & 14 \\ 0 & 0 & 3 & -5 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 3 & 0 & 0 & 6 \\ 0 & 3 & 0 & 19 \\ 0 & 0 & 3 & -5 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 19/3 \\ 0 & 0 & 1 & -5/3 \end{bmatrix} \end{aligned}$$

so  $x = \begin{bmatrix} 2 \\ 19/3 \\ -5/3 \end{bmatrix}$  is a least-squares solution.

(c) Again let  $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$  and  $b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$ .

Compute the orthogonal projection of  $b$  onto the **orthogonal complement** of the column space of  $A$ .

If  $x = \begin{bmatrix} 2 \\ 19/3 \\ -5/3 \end{bmatrix}$  is the least-squares solution in the previous part then

$$Ax = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

is the orthogonal projection of  $b$  onto the column space of  $A$ .

The orthogonal projection of  $b$  onto the orthogonal complement of the column space of  $A$  is therefore

$$b - Ax = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 3 \\ 3 \\ 0 \end{bmatrix}}$$

**Problem 8.** (5 + 10 = 15 points)

(a) Compute the singular values  $\sigma_1 \geq \sigma_2$  of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

The singular values of  $A$  are the square roots of the eigenvalues of

$$A^T A = \begin{bmatrix} 25 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since this matrix is diagonal, its eigenvalues are 25 and 2, so the singular values of  $A$  are

$$\sigma_1 = 5 \quad \text{and} \quad \sigma_2 = \sqrt{2}$$

(b) Find a singular value decomposition for

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

In other words, find a  $4 \times 4$  invertible matrix  $U$  and a  $2 \times 2$  invertible matrix  $V$  with  $U^{-1} = U^T$  and  $V^{-1} = V^T$  such that

$$A = U\Sigma V^T \quad \text{where} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let  $\sigma_1 = 5$  and  $\sigma_2 = \sqrt{2}$  be the singular values of  $A$ .

Since  $A^T A$  is diagonal, the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are an orthonormal basis of eigenvectors.

$$\text{Next define } u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} \quad \text{and} \quad u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Adding the vectors

$$u_3 = \frac{1}{\sigma_1} A v_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \end{bmatrix} \quad \text{and} \quad u_4 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

gives an orthonormal basis  $u_1, u_2, u_3, u_4$  or  $\mathbb{R}^4$ .

The desired matrices  $U$ ,  $\Sigma$ , and  $V$  are then

$$U = \begin{bmatrix} 3/5 & 0 & 3/5 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 4/5 & 0 & -4/5 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 5 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$