TLDR

Quick summary of today's notes. Lecture starts on next page.

- A *linear equation* in variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

  where the a_i's and b are numbers. Like

$$4x_1 + 5x_2 - \pi x_3 = -1 \quad \text{or} \quad x_1 = x_2 + 3 \quad \text{or} \quad 0 = 0 \quad \text{or even} \quad 0 = 1,$$

  but NOT $x_1^2 + x_2^2 = 1$ or $|x_1| + 3x_2 = 0$ or $\sin(x_1) = \sqrt{2}/2$ or $2^{x_1+x_2} = 4$.

- A *linear system* is a list of linear equations.
- A *solution* to a linear system is a list of values we can assign the variables that make all equations in the system true. The linear system with two equations $x_1 + x_2 = 7$ and $x_2 - x_1 = 1$ has a solution given by $(x_1, x_2) = (3, 4)$. Two linear systems are *equivalent* if they have the same solutions.
- Any linear system has 0, 1, or infinitely many solutions. If a linear system has two different solutions then it has infinitely many. If a linear system has no solutions then it is called *inconsistent*.
- A *matrix* is a rectangular array of numbers like $\begin{bmatrix} \sqrt{2} \end{bmatrix}$ or $\begin{bmatrix} 1.1 & -1.1 \\ 1.1 & 1.2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 7 & -1 \\ 0 & 4 & 3 \end{bmatrix}$.
- A matrix with m rows and n columns is said to be "$m \times n$" or "m-by-n."
- There are two important matrices associated to a linear system: the *coefficient matrix* and the *augmented matrix*. These are best defined by example:

$$\underbrace{\begin{aligned} 3x_1 + x_3 &= 8 \\ x_2 - x_3 &= 0 \\ 5x_1 + 4x_2 + 2x_3 &= 1 \end{aligned}}_{\text{linear system}} \rightsquigarrow \underbrace{\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 5 & 4 & 2 \end{bmatrix}}_{\text{coefficient matrix}} \quad \text{and} \quad \underbrace{\begin{bmatrix} 3 & 0 & 1 & 8 \\ 0 & 1 & -1 & 0 \\ 5 & 4 & 2 & 1 \end{bmatrix}}_{\text{augmented matrix}}.$$

- There are three *(elementary) row operations* we can perform on a matrix:

  (1) add a multiple of one row to another row,

  (2) multiply a row by a *nonzero* number,

  (3) swap two rows.

  For example:

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{row op. (1)}} \begin{bmatrix} 10 & 20 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{row op. (2)}} \begin{bmatrix} -5 & -10 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{row op. (3)}} \begin{bmatrix} 1 & 2 \\ -5 & -10 \end{bmatrix}.$$

  Two matrices are *row equivalent* if one can be transformed to the other by applying a finite sequence of these row operations.

- Linear systems with row equivalent augmented matrices have the same solutions (are *equivalent*).

1 Introduction

Check the course website

`http://www.math.ust.hk/~emarberg/teaching/2020/Math2121/`

for the syllabus and other course details.

Each lecture corresponds to one or more sections in the textbook.

Today's lecture corresponds to Section 1.1.

For a more detailed discussion of the topics in any particular lecture, see the textbook.

Throughout, we'll be using the following notation:

- $\mathbb{R}$ denotes the real numbers.
- $\mathbb{Q}$ denotes the rational numbers p/q.
- $\mathbb{Z}$ denotes the integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{N}$ denotes the nonnegative integers $\{0, 1, 2, \dots\}$.

Ellipsis ("...") notation: we write $a_1, a_2, \dots, a_7$ instead of the full list $a_1, a_2, a_3, a_4, a_5, a_6, a_7$.

We use the same convention to write $a_1, a_2, \dots, a_n$ even when n is a variable integer.

2 Systems of linear equations

Let $x_1, x_2, \dots, x_n$ be variables, where $n \geq 1$ is some integer.

Let $a_1, a_2, \dots, a_n, b$ be numbers in $\mathbb{R}$.

Unlike in calculus, where our favorite variables are x, y, z, in linear algebra we prefer $x_1, x_2, x_3, \dots$ since later we will want to go beyond 3 dimensions.

Definition. We refer to

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

as a *linear equation* in the variables $x_1, x_2, \dots, x_n$.

Notation. Another way of writing this equation is $\sum_{i=1}^{n} a_ix_i = b$.

The symbol "$\sum$" is the Greek letter sigma, for "sum."

There are many other equivalent ways of writing the same equation. For example:

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n - b = 0, \\ b = a_1x_1 + a_2x_2 + \cdots + a_nx_n, \\ a_1x_1 + a_2x_2 + a_3x_3 = b - a_4x_4 - a_5x_5 - \cdots - a_nx_n, \end{aligned}$$

and so forth. We consider all of these equations to be the same thing.

Example. The following are all linear equations in the variables x_1, x_2, x_3:

$$3x_1 = 2x_2, \qquad 3x_1 + \tfrac{4}{3}x_2 - \sqrt{2}x_3 = 7, \qquad 0 = 0, \qquad 0 = 1.$$

Even though the last two equations involve no variables, they have the form required of a linear equation. (The last equation is false, but a false equation is still an equation.)

The following are *not* linear equations in the variables x_1, x_2, x_3:

$$3x_1^2 + 4x_2 = 7, \qquad x_1 x_2 = x_3, \qquad 2^{x_1} = x_2, \qquad \sqrt{x_1^2} = x_2.$$

The last equation almost looks linear, but remember that $\sqrt{x_1^2} = |x_1|$.

A *system of linear equations* or *linear system* is a list of linear equations.

Example.

$$\begin{aligned} 2x_1 - x_2 + \sqrt{3}x_3 &= 8 \\ x_1 - 4x_3 &= 8 \\ x_2 &= 0 \end{aligned}$$

is a linear system in the variables x_1, x_2, x_3.

When we discuss a linear system, there is always a set of associated variables, usually $x_1, x_2, \ldots, x_n$ for some n. As we see in the preceding example, not every equation needs to involve every variable x_i. In fact, it could happen the some variable x_i appears in none of the equations.

Definition. A *solution* of a linear system in variables $x_1, x_2, \ldots, x_n$ is a list of n numbers $(s_1, s_2, \ldots, s_n)$ with the property that if we set $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$ in our equations, we get all true statements.

A solution $(s_1, s_2, \ldots, s_n)$ is *nonzero* if at least one number $s_i \neq 0$.

Two linear systems are *equivalent* if they have the same set of solutions.

Example. How many solutions can a linear system have?

1. The system

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$

   has one solution $(s_1, s_2) = (3, 2)$.

2. But the system

$$\begin{cases} x_1 - 2x_2 = -1 \\ 3x_1 - 6x_2 = -3 \end{cases}$$

   has many solutions: $(s_1, s_2) = (1, 1)$ or $(3, 2)$ or $(5, 3)$ or $\ldots$.

3. Whereas the system

$$\begin{cases} x_1 - 2x_2 = -1 \\ x_1 - 2x_2 = 0 \end{cases}$$

   has no solutions.

Theorem. A linear system in two variables x_1 and x_2 has either 0, 1, or infinitely many solutions.

Proof by geometry. A solution to one equation $ax_1 + bx_2 = c$ represents a point on a line after we identify the pair of numbers (x_1, x_2) with a point in the Cartesian plane.

A solution to a system of 2-variable linear equations represents a point where the lines defined by the equations all intersect.

But a collection of lines all intersect at either 0 points (they don't have a common intersection), 1 point (the unique point of intersection) or at infinitely many points (in the case when the lines are all *the same line*, though they might come from different equations). □

Proof by algebra. We just need to check that if our linear system has two different solutions, then it has infinitely many solutions.

Given a linear system, define the associated *homogeneous system* to be the linear system in which each equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is replaced by $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

For example the homogeneous system of

$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases} \quad \text{is} \quad \begin{cases} x_1 - 2x_2 = 0 \\ -x_1 + 3x_2 = 0. \end{cases}$$

If (s_1, s_2) is a solution to our starting system and (t_1, t_2) is a solution to the associated homogeneous system, then $(s_1 + t_1, s_2 + t_2)$ is also a solution to our starting system.

This is because if $a_1s_1 + a_2s_2 = b$ and $a_1t_1 + a_2t_2 = 0$ then

$$a_1(s_1 + t_1) + a_2(s_2 + t_2) = (a_1s_1 + a_2s_2) + (a_1t_1 + a_2t_2) = b + 0 = b.$$

On the other hand, if the homogeneous system has a nonzero solution (t_1, t_2), then it has infinitely many solutions: the pairs $(2t_1, 2t_2)$, $(3t_1, 3t_2)$, $(4t_1, 4t_2)$, and so on are all solutions, since for example

$$\text{if } a_1t_1 + a_2t_2 = 0 \text{ then } a_1(4t_1) + a_2(4t_2) = 4(a_1t_1 + a_2t_2) = 4 \cdot 0 = 0.$$

Combining these observations means that if our starting system has a solution and the homogeneous system has a nonzero solution, then the starting system has infinitely many solutions.

Finally, observe that if our starting system has two different solutions (s_1, s_2) and (r_1, r_2), then

$$(t_1, t_2) = (s_1 - r_1, s_2 - r_2)$$

is a nonzero solution to the homogeneous system, since if $a_1s_1 + a_2s_2 = b$ and $a_1r_1 + a_2r_2 = b$ then

$$a_1(s_1 - r_1) + a_2(s_2 - r_2) = (a_1s_1 + a_2s_2) - (a_1r_1 + a_2r_2) = b - b = 0.$$

□

A linear system is *consistent* if it has one or infinitely many solutions, and *inconsistent* if it has zero solutions. Both the algebraic and geometric proofs generalize to any number of variables. (Think about how to do this!) Therefore:

Theorem. A linear system is either consistent or inconsistent, and therefore has either 0, 1, or infinitely many solutions.

3 Matrices

A *matrix* is just a rectangular array of numbers, like these ones:

$$\begin{bmatrix} 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 5 & 3 \\ 2 & \pi \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & 6 & 4 & 3 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

We denote a general matrix by

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$$

Here “A_{23}” is pronounced “A, two, three”. This matrix is 3-by-4: it has 3 rows and 4 columns.

One says that a matrix A is m-by-n or $m \times n$ if it has m rows and n columns.

We usually write A_{ij} (pronounced "A, i, j") for the entry in the ith row and jth column of a matrix A.

Matrices are useful as a compact way of writing a linear system.

Consider the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

Define the *coefficient matrix* of this system to be

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

In other words, the matrix A where A_{ij} is the coefficient of x_j in the ith equation.

The *augmented matrix* of the system is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \quad \text{which is sometimes written as} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right].$$

We consider both of these to be the same matrix. The | on the right is just there to remind us that this is an augmented matrix rather than a coefficient matrix.

Exercise: how would you generalize this definition to any linear system?

4 Solving linear systems

We solve linear systems by adding equations together to cancel variables.

Example. To solve

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

we first add -5 time equation 1 to equation 3 to get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right].$$

We then multiply equation 2 by 1/2 to get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{array}\right].$$

We then add -10 times equation 2 to equation 3:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 30x_3 &= -30 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{array}\right].$$

Multiple equation 3 by 1/30:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= -1 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

The augmented matrix of the last system if *triangular*: all entries in positions (i, j) with $i > j$ are zero.

Remember that i is the row, j is the column.

We can easily solve for x_1, x_2, x_3 from a triangular system, working from the bottom up:

- The last equation $\boxed{x_3 = -1}$ is already as simple as possible.
- Substitute into second equation: $x_2 - 4x_3 = x_2 - 4(-1) = 4 \Rightarrow \boxed{x_2 = 0}$.
- Substitute into first equation: $x_1 - 2x_2 + x_3 = x_1 - 2(0) + (-1) = 0 \Rightarrow \boxed{x_1 = 1}$.

Definition. In solving this system of equations, we performed the following *(elementary) row operations* on the augmented matrix of the system:

1. Replacement: replace one row by the sum of itself and a multiple of another row:
$$\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 501 & 602 \\ 5 & 6 \end{bmatrix}.$$
2. Scaling: multiply all entries in a row by a *nonzero* number: $\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ -500 & -600 \end{bmatrix}$.
3. Interchange: swap two rows: $\begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$.

Two matrices are *row equivalent* if one can be transformed to the other by a sequence of row operations.

Each row operation is reversible. (Exercise: why?)

Theorem. If the augmented matrices of two linear systems are row equivalent, then the systems are equivalent (i.e., have same solutions).

Proof. Here's the idea, minus the details: check that performing one row operation does not change whether a given $(s_1, s_2, \dots, s_n)$ is a solution to the linear system. □

Given a linear system with augmented matrix A, suppose we perform row operations on A until we get a matrix T with the property that whenever T_{ij} is the first nonzero entry in the ith row of T going left to right, then T_{ij} is the last nonzero entry in the jth column of T going top to bottom. For example:

$$T = \begin{bmatrix} 1 & 6 & 8 & 9 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \qquad \text{or} \qquad T = \begin{bmatrix} 1 & 6 & 8 & 9 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

From T in this form, we can easily determine if the system we started out with is consistent or inconsistent.

If T is the left matrix, the system is consistent: we have

$$x_4 = 4, \qquad 3x_3 + 2x_4 = 1, \qquad \text{and} \qquad x_1 + 6x_2 + 8x_3 + 9x_4 = 0.$$

Exercise: find a solution!

If T is the right matrix, the system is inconsistent: it includes the false equation $0 = 2$ in the last row.

In general, a linear system is inconsistent if and only if its augmented matrix can be transformed by row operations to a matrix with a row of the form $\begin{bmatrix} 0 & 0 & \dots & 0 & q \end{bmatrix}$ where $q \neq 0$. We'll prove this next time, after introducing the course's most important algorithm, *row reduction to echelon form*.

5 Vocabulary

Keywords from today's lecture:

1. **Linear equation**.

   An equation of the form $a_1x_1 + a_2x_2 + \dots a_nx_n = b$ where n is a positive integer, $a_1, a_2, \dots, a_n, b$ are numbers, and $x_1, x_2, \dots, x_n$ are variables.

   Example: $3x_1 - \frac{1}{7}x_3 = x_4 + 5$.

2. **Linear system** or **system of linear equations**.

   A list of one or more linear equations.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$

3. **Solution** to a linear system.

   A solution to one linear equation $a_1x_1 + a_2x_2 + \dots a_nx_n = b$ is a list of numbers $(s_1, s_2, \dots, s_n)$ such that $a_1s_1 + a_2s_2 + \dots a_ns_n$ is equal to b. A solution to a linear system is a list of numbers that is simultaneously a solution to every equation in the system.

   Example: a solution to $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$ is $(s_1, s_2) = (\frac{7}{4}, \frac{5}{4})$.

4. **Equivalent** linear systems.

   Two linear systems with the same sets of variables and same sets of solutions.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$ and $\begin{cases} 2x_1 + 2x_2 = 6 \\ x_1 - 3x_2 + 2 = 0 \end{cases}$ are equivalent.

5. **Consistent** linear system.

   A linear system with at least one solution.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 3x_2 - x_1 = 2 \end{cases}$ is consistent.

6. **Inconsistent** linear system.

   A linear system with no solutions.

   Example: $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 4 \end{cases}$ is inconsistent.

7. **Matrix**.

   A rectangular array of numbers. A matrix A is $m \times n$ if it has m rows and n columns.

   We write A_{ij} for the entry of A is row i and column j.

   Example: $A = \begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix}$. This matrix is 2×3 and $A_{21} = \sqrt{2}$ while $A_{12} = -1$.

8. **Coefficient matrix** of a linear system.

   For a linear system m equations with n variables, the $m \times n$ matrix that records the coefficients of the variables.

   Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix}$ is the coefficient matrix of $\begin{cases} -x_2 + 2x_3 = 3 \\ \sqrt{2}x_1 + 5x_2 + 6x_3 = 7 \end{cases}$.

9. **Augmented matrix** of a linear system.

   For a linear system m equations with n variables, the $m \times (n+1)$ matrix that records the coefficients of the variables and the constant on the other side of each equation.

   Example: $\begin{bmatrix} 0 & -1 & 2 & 3 \\ \sqrt{2} & 5 & 6 & 7 \end{bmatrix}$ is the augmented matrix of $\begin{cases} -x_2 + 2x_3 = 3 \\ \sqrt{2}x_1 + 5x_2 + 6x_3 = 7 \end{cases}$.

10. **Elementary row operator** on a matrix.

    One of the following operations on a matrix: replace one row by the sum of the row and a multiple of another row, multiply all entries in row by a fixed number, or swap two rows.

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{2} & 9 & 14 \\ \sqrt{2} & 5 & 6 \end{bmatrix}$

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 2 \\ 5\sqrt{2} & 25 & 30 \end{bmatrix}$

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} \sqrt{2} & 5 & 6 \\ 0 & -1 & 2 \end{bmatrix}$.

11. **Row equivalent matrices**.

    Matrices that can be transformed to each other by a sequence of row operations.

    Example: $\begin{bmatrix} 0 & -1 & 2 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{2} & 9 & 14 \\ \sqrt{2} & 5 & 6 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{2} & 9 & 14 \\ 5\sqrt{2} & 25 & 30 \end{bmatrix} \to \begin{bmatrix} 5\sqrt{2} & 25 & 30 \\ 2\sqrt{2} & 9 & 14 \end{bmatrix}$