

**Instructions:** Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.

Due on **Thursday, March 24.**

Except when mentioned otherwise, all Lie algebras and vector spaces below are finite-dimensional and defined over an algebraically closed field  $\mathbb{F}$  of characteristic zero.

1. Compute the basis of  $\mathfrak{sl}_n(\mathbb{F})$  which is dual to the standard basis via the Killing form  $\kappa$ . (This is an upgrade of an earlier exercise which considered the case  $n = 2$ .)

Use this to compute  $t_\alpha$  and  $h_\alpha := 2t_\alpha/\kappa(t_\alpha, t_\alpha)$  for each  $\alpha \in \Phi$ , relative to the root space decomposition in which  $H \subset \mathfrak{sl}_n(\mathbb{F})$  is the maximal toral subalgebra of traceless diagonal matrices. Recall that  $t_\alpha \in H$  for  $\alpha \in H^*$  is the unique element with  $\kappa(t_\alpha, h) = \alpha(h)$  for all  $h \in H$ .

2. Let  $L$  be a semisimple Lie algebra with a maximal toral subalgebra  $H$ . Prove that if  $h \in H$  then the centralizer  $C_L(h) := \{X \in L : [X, h] = 0\}$  is a reductive Lie algebra. Prove that there are elements  $h \in H$  with  $C_L(h) = H$ . For which  $h \in H$  does this hold if  $L = \mathfrak{sl}_n(\mathbb{F})$  and  $H$  is the subalgebra of traceless diagonal matrices?

If  $\Phi \subset E$  and  $\Phi' \subset E'$  are root systems, then an isomorphism  $\Phi \rightarrow \Phi'$  is a linear bijection  $f : E \rightarrow E'$  with  $f(\Phi) = \Phi'$  and  $\langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle$  for all  $\alpha, \beta \in \Phi$ , where  $\langle \beta, \alpha \rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ . The map  $f$  does not need to be an isometry with respect to the forms  $(\cdot, \cdot)$  on  $E$  and  $E'$ .

3. Prove that every three dimensional semisimple Lie algebra has the same root system as  $\mathfrak{sl}_2(\mathbb{F})$  up to isomorphism, and so is isomorphic to  $\mathfrak{sl}_2(\mathbb{F})$ .
4. Prove that no four, five, or seven dimensional semisimple Lie algebras exist.

Below, let  $\Phi$  be a root system with ambient space  $E$ , associated form  $(\cdot, \cdot)$ , and Weyl group  $W$ .

5. Define  $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$  where  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ . Check that  $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$  for  $\alpha, \beta \in \Phi$ . Prove that  $\Phi^\vee$  is a root system (in the same ambient vector space) whose Weyl group is isomorphic to  $W$ . Draw a picture of  $\Phi^\vee$  versus  $\Phi$  when  $\Phi$  has type  $A_1, A_2, B_2$ , and  $G_2$ .
6. Let  $\Phi'$  be a nonempty subset of  $\Phi$  with  $-\Phi' = \Phi'$ , such that if  $\alpha, \beta \in \Phi'$  and  $\alpha + \beta \in \Phi$  then  $\alpha + \beta \in \Phi'$ . Prove that  $\Phi'$  is a root system in the subspace of  $E$  that it spans.
7. An *automorphism* of  $\Phi$  is an isomorphism of root systems from  $\Phi$  to itself. Prove that  $W$  is a normal subgroup of the group  $\text{Aut}(\Phi)$  of all automorphisms of  $\Phi$ .
8. Show by example that it can happen that  $\alpha - \beta \in \Phi$  for  $\alpha, \beta \in \Phi$  with  $(\alpha, \beta) \leq 0$ .