Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Thursday, March 31**.

Throughout, Φ denotes a root system in a real vector space E with Weyl group W. Fix a simple system Δ for Φ and define $\Phi^{\pm} \subset \Phi$ accordingly. Let $\ell : W \to \mathbb{N}$ denote the length function of W.

1. Define a map $\mathrm{sgn} : W \to \{\pm 1\}$ by $\mathrm{sgn}(w) = (-1)^{\ell(w)}$. Prove that this is a group homomorphism.
2. For $\gamma \in E$ let $P_\gamma = \{v \in E : (v, \gamma) > 0\}$. Prove for any finite set of linearly independent vectors $\gamma_1, \gamma_2, \ldots, \gamma_k \in E$ that the intersection $\bigcap_{i=1}^k P_{\gamma_i}$ is nonempty.
3. Prove that there is a unique element $w_0 \in W$ with $w_0(\Phi^+) = \Phi^-$. Prove that any reduced expression for w_0 must involve every simple reflection r_α for $\alpha \in \Delta$.
4. Recall that $\Phi^\vee = \{\alpha^\vee : \alpha \in \Phi\}$ where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Prove that if Φ is irreducible then so is $\Phi^\vee$.
5. Prove that if $0 \neq \alpha \in E$ and the reflection r_α belongs to W, then $r_\alpha = r_\beta$ for some $\beta \in \Phi$.
6. Prove that W is isomorphic to the direct product of the respective Weyl groups of the irreducible components of Φ.